Int. J. Heat Mass Transfer. Vol. 6, pp. 147–189. Pergamon Press 1963. Printed in Great Britain.

HEAT TRANSFER ACROSS TURBULENT, INCOMPRESSIBLE BOUNDARY LAYERS*

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Abstract—The paper contains a survey of the present status of knowledge concerning the transfer of heat by forced convection across incompressible turbulent boundary layers. The foundations of the semi-empirical theory are examined from first principles and the limitations of the theory are carefully noted. Elementary theories are described and an outline of D. B. Spalding's mathematically exact theory is given. The limiting cases of very high and very low Prandtl numbers are discussed. Finally, an outline of W. V. R. Malkus' theory of turbulent processes is sketched.

A conscientious attempt has been made to clarify all physical assumptions, to identify the major fundamental problems which require attention and to indicate directions in which the semi-empirical theory can be extended.

NOMENCLATURE

A,	empirical coefficient;
A_q,	k_t/c_p;
A_t,	μ_t/ρ;
B,	empirical coefficient;
C_1, C_2,	constants of integration;
H,	ratio, boundary layer displacement thickness to momentum thickness;
K,	empirical coefficient;
R,	Loitsianskii parameter; reference length;
T,	temperature;
U,	local free stream velocity;
a,	thermal diffusivity;
c_p,	specific heat at constant pressure;
$\mathbf{c}$,	velocity vector;
d,	exponent in viscosity–temperature relations;
g_x,	x-component of gravitational field;
k,	thermal conductivity of fluid;
l,	$(x - x_o)$, positive only;
n,	coefficient;
p,	pressure;
$p(y/\delta)$,	law of the wake expression;
$\dot{q}$,	heat flux;
r,	radius;
t,	time co-ordinate;
u, v, w,	velocity components in directions of co-ordinates x, y, z;
v_*,	friction velocity $v_* = \sqrt{(\tau_w/\rho)}$;
x, y, z,	orthogonal curvilinear co-ordinates.

Greek symbols

α,	empirical constant;
β,	heat flux parameter ($\beta = \dot{q}_w v_* / c_p T_w \tau_w$); also, coefficient of thermal expansion (Section 19);
γ,	incomplete gamma function;
Γ,	gamma function;
δ,	boundary layer thickness; (unsubscripted) velocity boundary layer thickness;
Δ,	distance from surface greater than δ and δ_T;
ϵ^+,	ν_e/ν;
η,	similarity parameter $\eta = (y^+)^3 Pr/9x^+$;
θ,	$T_w - T$;
κ,	coefficient;
λ,	wavelength;
μ,	dynamical viscosity;
$\Delta\mu$,	maximum change in μ across a boundary layer;
ν,	kinematic viscosity;
Π,	pressure gradient function;
ρ,	fluid density;
τ,	shear stress in fluid;
Φ,	dissipation function;
ψ,	stream function.

* Prepared for the International Colloquium on Turbulence, 28 August–2 September 1961, Marseilles, France. A brief outline is to be published in the Colloquium Proceedings.

Dimensionless parameters

c_f,	local coefficient of friction;
l^+,	$\int_0^l [v_*(x)/\nu]\,dx$;
u^+,	$\bar{u}/v_*$;
x^+,	xv_*/ν;
y^+,	yv_*/ν;
v_*,	$\sqrt{(\tau_w/\rho)}$;
ϵ^+,	ν_e/ν;
Ec,	Eckert number;
Nu,	Nusselt number;
Pr,	Prandtl number;
Re_x,	Reynolds number, based on x;
Ri,	Richardson number;
Sp,	Spalding function;
St,	Stanton number.

Subscripts

b,	buffer layer;
e,	effective;
l,	laminar (layer);
o,	property values relative to those of air; x_o, hydrodynamic starting length;
r,	with a rough wall;
s,	in a fluid stream;
t,	turbulent; δ_t, outer limit of law of the wall;
T,	thermal;
w,	at the surface;
x,	value based on x;
∞,	value deep upstream and cross-stream.

Superscripts

$+$,	dimensionless;
$'$,	fluctuating component;
$^-$,	arithmetic time average.

1. INTRODUCTION

THE MAIN objective of a theory of convective heat transfer is to permit the calculation of the heat flux $\dot{q}$, i.e. the rate of flow of heat per unit area and time transferred between a solid surface and a fluid stream in contact with it. It is desirable that such a calculation should be possible for any specified surface under any boundary and initial conditions.

This survey is restricted to the study of forced convection where the flow field consists of a two-dimensional turbulent boundary layer adjacent to a surface whose temperature, T_w, differs from that of the fluid stream, T_s, at some distance from the surface, and where the density variations of the fluid can be disregarded, whether they are caused by pressure differences or by thermal expansion.

The corresponding problem involving laminar flow can be reduced, from first principles, to a set of three simultaneous, partial differential equations for the velocity components u, v, and for the temperature difference

$$\theta = T_w - T. \tag{1.1}$$

Given the boundary and initial conditions for the flow and for the temperature fields, and given the properties of the fluid, it is only necessary to find mathematical methods, exact or approximate, in order to determine the local heat flux at the wall, which is then given by

$$\dot{q}_w(x, y, t) = k\left(\frac{\partial\theta}{\partial y}\right)_w, \quad \text{(at wall)}. \tag{1.2}$$

Hence the difficulties are exclusively of a mathematical nature.

In the study of turbulent convection a tractable mathematical formulation rests on a number of assumptions, some of which are heuristic and have not been confirmed by experiment. Consequently, it is possible to suggest several alternative analytic formulations. The complete reduction of the problem to explicit mathematical terms which are nearly, but not quite, equivalent to the formulation of the laminar problem has only been achieved very recently, notably by Spalding [1]. In this paper, we shall review the main developments which have resulted in the above formulation of the problem. It will, however, be found that several crucial questions of a physical nature must be answered before the formulation can be accepted.

The first observation of a region within a fluid, close to the surface of a body, which offers resistance to the transfer of heat was made by Péclet in 1844 [2]. Reynolds [3] recognized that an understanding of local conditions within the flow field would provide the key for an analytical theory of the transfer of heat. He also realized that in turbulent flows there must exist an intimate relation between the local shearing stress and the local heat flux since both depend upon the same basic mechanism. Prandtl [4] and Taylor [5] independently realized that the ideas propounded by Reynolds, coupled with the recognition of the boundary layer character of the temperature field as well as the velocity field, could provide a means for determining local rates of heat transfer. This approach was developed and refined by several workers, as is described in Sections 12 and 13. The application of these developments to the calculation of heat transfer across turbulent boundary layers is due to Spalding [1].

Before significant analytical developments were made, a systematic correlation of experimental data by use of the principles of dimensional analysis and physical similarity was begun by Nusselt [6] and extended by many others. Ignoring the details of the structure of flow patterns, these correlations allow mean heat transfer rates from surfaces to be estimated for many cases of practical importance. They lead to a lucid classification of numerous problems of considerable complexity and greatly simplify the task of performing experiments. An organized collection of such correlations may be found in the compendium of McAdams [7].

2. THE CENTRAL PROBLEM IN HEAT TRANSFER

As outlined in the introduction, the central problem in the analytic theory of heat transfer

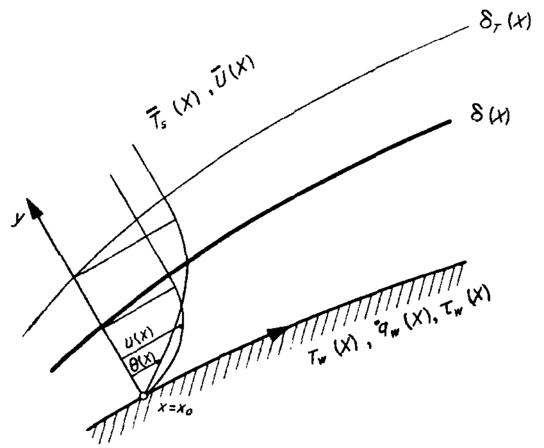

FIG. 1. The central problem in heat transfer.

can be formulated as follows (Fig. 1). Given a cylindrical surface S with a prescribed temperature distribution $T_w(x)$ over which there exists a velocity boundary layer $\delta(x)$ created by an external stream $U(x)$ and a thermal boundary layer $\delta_T(x)$ created by the temperature distribution $\bar{T}_s(x)$ in the free stream, and given the velocity as well as the temperature distribution, $u(y)$ and $\theta(y)$ at some section $x = x_o$, determine the total temperature field $\theta(x, y)$, and hence the heat flux $\dot{q}_w(x)$ from (1.2).* In the present paper attention is restricted exclusively to cases when the resulting boundary layer is turbulent but incompressible. Consequently, all preceding quantities constitute time averages.

It is immediately apparent that the local heat flux, (1.2), is analogous in form to the local shearing stress

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w \quad \text{(at wall)} \tag{2.1}$$

at the wall, and the consideration of the relations between them will prove very useful.

The problem of heat transfer is never treated with all possible generality. In all cases considered, (Fig. 2), the approaching stream is one of uniform velocity $\bar{U}_\infty$ and uniform temperature $\bar{T}_\infty$. Hence, the free-stream temperature along the edge of the thermal boundary layer remains constant, and all the heat transferred from the surface to the stream remains within the thermal boundary layer. Under these circumstances the heat flux at the edge is no longer present and the only heat transported across the edge is carried by the fluid entrained into the growing velocity boundary layer.

With respect to the wall temperature, it is possible to discern four typical problems. In the simplest problem, Fig. 2a, the surface temperature $\bar{T}_w(x)$ remains constant, but often, Fig. 2b, actual conditions more nearly approximate a constant heat flux $\dot{q}_w$. In the most general case, Fig. 2c, the surface temperature $\bar{T}_w(x)$ is prescribed, but the special case, Fig. 2d, when it undergoes a discontinuous jump, is of importance, because from its solution it is possible to

* This is, of course, identical for a turbulent and a laminar boundary layer because in the layer immediately adjacent to the wall heat can only be transferred by molecular conduction.

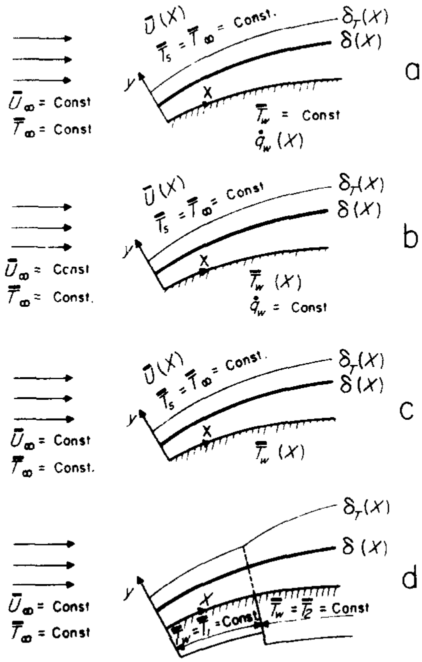

FIG. 2. Geometries of the typical problems.

construct solutions for arbitrary temperature distributions.

The imposition of a temperature difference between the solid wall and the fluid flowing past it gives rise to two new effects. First, it creates a temperature field whose detailed appearance determines the local heat flux. Secondly, it affects the properties of the fluid, since they all depend on temperature. The changes in the properties due to variations in temperature, notably the changes in viscosity, modify the flow field which now becomes different from the corresponding isothermal field. Finally, the consideration of the new parameters associated with the temperature field brings into play additional properties of the fluid, notably its Prandtl number, and differences in the behavior of different fluids are strongly accentuated.

The relation between the thicknesses of the two boundary layers, $\delta(x)$ and $\delta_T(x)$, is a quantity which is ultimately determined by the analysis. It is, however, useful to remember that this relation depends as much on the temperature and velocity distributions, $\theta_s(x)$ and $\bar{U}(x)$, as on the properties of the fluid, in particular on its Prandtl number, Pr. Of particular importance

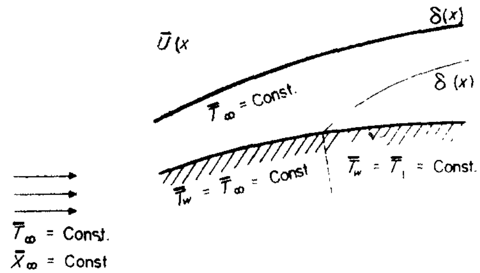

FIG. 3. Thermal entry length.

are problems involving a thermal entry length, Fig. 3, in which the thermal boundary layer develops from a definite point onwards, the velocity boundary layer being already fully established. In such cases, the wall temperature remains constant and equal to the free-stream temperature $\bar{T}_\infty$ over a certain length, changing its value suddenly beyond this length, thus initiating the growth of a thermal boundary layer which is at first much thinner than the velocity boundary layer. Other things being equal, the thermal boundary layer in a fluid of large Prandtl number, i.e. small conductivity, is much thinner than one in a fluid of small Prandtl number which corresponds to a large thermal conductivity.

Independently of the details of any particular problem, it is easy to prove that the following integral equation must apply:

$$\frac{\partial}{\partial x}\int_0^{\Delta} c_p\, u(\theta_s - \theta)\,\mathrm{d}y = \dot{q}_w \left[\text{or} = k\left(\frac{\partial\theta}{\partial y}\right)_w\right]. \quad (2.2)$$

It represents the energy balance for a transverse section of thickness $\mathrm{d}x$, extending from $y = 0$ to $y = \Delta$, where $\Delta > \delta$ and $\Delta > \delta_T$. For constant values of ρ and c_p, the equation can be written

$$\frac{\partial}{\partial x}\int_0^{\Delta} u(\theta_s - \theta)\,\mathrm{d}y = a\left(\frac{\partial\theta}{\partial y}\right)_w \left(\text{or} = \frac{\dot{q}_w}{\rho c_p}\right). \quad (2.3)$$

Corresponding with the main interest in heat transfer, most experimental investigations have concentrated on the measurement of surface heat fluxes. Precise measurements of temperature profiles within a fluid have rarely been made, and then only in air, e.g. [8, 9, 10, 11, 12, 13, 14]. The inaccessibility of the laminar sublayer which most critically affects the gradient $(\partial\theta/\partial y)_w$ at the wall makes it very difficult to determine heat fluxes from the gradient of the temperature profile itself. The low accuracy, modest extent and limited scope of these measurements has prevented them from becoming a starting point for an empirical theory of heat transfer.

With very few exceptions [15, 10, 11], all available experimental results pertaining to the transfer of heat across turbulent incompressible, as opposed to compressible, streams, is confined to flows through pipes, but these results can often be used to verify theories of heat transfer to boundary layers, in the same way as is done in the study of skin friction.

3. SURVEY OF THE THERMAL PROPERTIES OF FLUIDS

Before writing down the fundamental equations of motion, continuity and energy, it is useful to examine the conditions under which it will be permissible to assume that the properties of the fluid are constant throughout the field. Since the implications of assuming a constant density are well known, and since compressibility effects are excluded from the present survey, it is permissible to assume that the density, ρ, remains constant, except in cases when even small changes in density caused by thermal expansion give rise to buoyancy forces in a gravitational field. These are the effects which lead to natural convection. The remaining properties of interest in heat transfer include: the absolute viscosity μ, the kinematic viscosity, ν, the thermal conductivity, k, the thermal diffusivity, $a = k/\rho c_p$, the Prandtl number, Pr, and the specific heat at constant pressure, c_p. Representative values of these properties for gases and liquids have been listed in Tables 1 and 2. Table 1 contains values measured relative to air, at 1 atm, whereas Table 2 lists the ratios of the respective quantities for a specified change in temperature, also at 1 atm, and thus give an idea of the rate of variation of each property over a relatively narrow temperature range. It will be remembered that only the kinematic viscosity of gases is affected by pressure, all the other quantities being virtually insensitive to changes in it.

The data in Table 1 demonstrate the extremely wide range of values encountered in heat transfer problems. The viscosity varies by a factor of 10^5, the kinematic viscosity varies by a factor of 10^4, the thermal conductivity varies by a factor of 10^3, the range of Prandtl numbers is 10^6 to 1 and that of specific heats is 10^2 to 1. It is important to note that the range of variation is much smaller among gases, particularly as regards the Prandtl number, than it is among liquids. It is clear that experimental data on heat transfer cannot be confined to gases only and must span a wide range of substances.

The ratios listed in Table 2 show that the properties of liquids are very sensitive to temperature changes, much more so than those of gases. Thus the assumption of constant properties is more nearly justified in the case of gases, which are inherently compressible, than in the case of the inherently incompressible

Table 1. *Range of variation of properties of fluids at* 0°C *and* 1 atm (*relative to air; Prandtl number absolute*)

	μ_o	ν_o	k_o	a_o	c_{po}	Pr
Gases:						
Air	1	1	1	1	1	0·72
Hydrogen	0·50	7·1	7·3	7·4	14	0·71
Helium	1·1	8·0	6·3	0·86	5·2	0·70
Water vapor*	0·75	1·6	1·0	1·0	2·1	1·1
Liquids:						
Water	110	0·14	23	0·0068	4·2	14
Oil 1†	760	1·1	6·0	0·0046	1·8	170
Oil 2†	2200	2·7	5·2	0·0039	1·9	480
Oil 3†	47000	66	6·0	0·0046	1·8	10000
Mercury	100	0·0092	340	0·22	0·12	0·029

* at 100°C. † at 20°C.

Table 2. Variation of fluid properties with temperature. Ratio of property at 20°C to that at 100°C at 1 atm

	$\frac{\mu_{100}}{\mu_{20}}$	$\frac{\nu_{100}}{\nu_{20}}$	$\frac{k_{100}}{k_{20}}$	$\frac{a_{100}}{a_{20}}$	$\frac{c_{p100}}{c_{p20}}$	$\frac{Pr_{100}}{Pr_{20}}$
Gases:						
Air	1·2	1·5	1·2	1·5	1·0	1·0
Hydrogen	1·2	1·5	1·2	1·6	1·0	0·96
Helium	1·2	1·5	1·1	1·4	1·0	1·0
Water vapor*	1·3	1·7	1·4	1·9	0·90	0·87
Liquids:						
Water	0·36	0·30	1·1	1·2	1·0	0·25
Oil 1	0·22	0·16	0·97	0·88	1·2	0·19
Oil 2	0·098	0·10	0·95	0·84	1·2	0·13
Oil 3	0·021	0·025	0·94	0·85	1·2	0·026
Mercury	0·82	0·81	1·2	1·3	1·0	0·56

* Between 200°C and 100°C.

liquids. Comparison between experiment and a theory in which the properties have been assumed constant is not always conclusive, and requires suitable means for determining the appropriate average values of the varying properties.

It is useful to note that temperature differences which correspond to a given heat flux are smaller in turbulent than in laminar convection, and consequently, in practical problems the assumption of constant properties is more justified in turbulent than in laminar convection.

Although the variation in specific heat is of the same order as that in thermal conductivity, it is usual to neglect it.

4. FUNDAMENTAL ASSUMPTIONS. GENERAL CONDITIONS OF SIMILARITY

In convective heat transfer, as in fluid dynamics, it is now generally accepted that the entire behavior of the flow in all regions is implied in the Navier–Stokes equations, together with the equations of continuity and conservation of energy. In heat transfer the solution of the energy equation is the prime concern.

Turbulent flow is conceived as developing from laminar flow by the growth of unstable disturbances which originate from the boundaries of the fluid. Even when everything is done to produce a steady flow, once turbulence is established the flow is inherently non-steady, and the local parameters fluctuate about their mean values. Therefore, even in the case of a flow in which the mean values are constant, the time derivative must be retained in the fundamental equations.

Conservation of momentum

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) + \left(\frac{\partial \mu}{\partial x} \cdot \frac{\partial u}{\partial x}\right) + \left(\frac{\partial \mu}{\partial y} \cdot \frac{\partial u}{\partial y}\right) + \left(\frac{\partial \mu}{\partial z} \cdot \frac{\partial u}{\partial z}\right), \quad (4.1)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z};$$

there being two similar equations for velocity components v and w.

Conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (4.1a)$$

Conservation of energy

$$\varrho c_p \frac{D\theta}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial \theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial \theta}{\partial z}\right) + \mu\Phi \quad (4.1b)$$

where

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \\ + \left(\frac{\partial y}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial y}{\partial z}\right)^2 + \\ + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$$

denotes the dissipation function. The only restrictions imposed upon these equations are that the density is constant, and that the enthalpy is independent of pressure.

The full equations cannot be considered for further analysis in all their generality, owing to their complexity, and to our inability to postulate in them realistic, time-dependent boundary and initial conditions. However, two important observations can be made from them. First, the flow field depends upon the variation of viscosity with position, and therefore with temperature, so that in general the two fields are coupled. Secondly, the energy equation is similar to the x-component of the momentum equation, and it is possible that the temperature field can become similar to the velocity field under certain conditions. The terms which preclude strict similarity arise from the pressure gradient $\partial p/\partial x$, from the dissipation function Φ and from the fact that the viscosity μ and the thermal conductivity k are described by different functions of temperature.

If the further, and considerably restricting, assumption of constant properties is made, the continuity equation remain unaltered, whilst the equations for the conservation of momentum and of energy become

$$\frac{D\mathbf{c}}{Dt} = -\frac{1}{\rho}\,\mathrm{grad}\,p + \nu\nabla^2\mathbf{c} \tag{4.2a}$$

$$\frac{D\theta}{Dt} = a\,\nabla^2\theta + \frac{\nu}{c_p}\,\Phi \tag{4.2b}$$

From these equations it is clear that under the assumption of constant properties the velocity field becomes entirely independent of the temperature field. The flow field, being determined regardless of any heat transfer, can be studied independently. In the theory of heat transfer the velocity field is usually assumed to be identical with that which would exist in the absence of the actual temperature field, and it must not be forgotten that this involves a serious approximation.

On the other hand, with this simplification the velocity field u and the temperature field θ can become similar when the pressure gradient $\partial p/\partial x$ and the dissipation function Φ become small everywhere compared with the remaining terms. The former condition is satisfied on a flat plate at zero incidence and nearly so on cylinders of gentle curvature, whereas the latter condition is satisfied when the viscosity is not too large. More precisely, by performing a standard estimation of terms [16, p. 296], it is easy to show that this is the case when the product of the Prandtl and Eckert numbers is very small, i.e. when

$$Pr\,.\,Ec = \frac{\mu U_\infty^2}{k\theta_\infty} \ll 1. \tag{4.3}$$

Thus we compare the equations

$$\frac{Du}{Dt} = \nu\nabla^2 u \tag{4.4a}$$

$$\frac{D\theta}{Dt} = a\nabla^2\theta \tag{4.4b}$$

and note that they will be similar when

$$Pr = \frac{\nu}{a} = 1. \tag{4.5}$$

Their solutions will be similar if their corresponding boundary conditions are similar. Owing to the omission of the pressure gradient, strict similarity can only exist on a flat plate, when the velocity and temperature profiles are identical at some instant at some cross section, and when the temperature of the plate is constant.

In spite of these drastic restrictions, the concept of similarity between the two fields plays an important part in heat transfer, but it is emphasized that no similarity can be expected to exist in problems involving thermal entry lengths. The preceding argument only established sufficient conditions for similarity. A systematic search for conditions which are both necessary and sufficient for similarity to exist does not seem to have been made. It must, however, be remembered that in steady-state problems no boundary conditions in time can be fixed. This makes it necessary to investigate them separately, and to deduce them from observation, as will be explained more fully after equation (4.9b) and in Section 10.

The impossibility of deriving turbulent flow patterns, and hence temperature fields, directly from the full equations is now clearly recognized, and it is fruitful to form time-averages of the fluctuating quantities in the conservation equations to develop a physical insight into the phenomena involved. Retaining the assumption of constant properties, it is further assumed that the independent variables in the equations can be represented, for a quasi-steady stream, as an average value (denoted by a superscript bar) and a fluctuating component (denoted by a prime), the time average of every fluctuating component converging to zero for large time intervals. Thus

$$\begin{aligned} u &= \bar{u} + u'; & \overline{u'} &= 0 \\ p &= \bar{p} + p'; & \overline{p'} &= 0 \\ \theta &= \bar{\theta} + \theta'; & \overline{\theta'} &= 0, \text{ etc.} \end{aligned} \qquad (4.6)$$

It can be seen that the additional terms in the energy equation

$$\frac{\partial}{\partial x}(\overline{\theta' u'}), \quad \frac{\partial}{\partial y}(\overline{\theta' v'}), \quad \frac{\partial}{\partial z}(\overline{\theta' w'}), \qquad (4.7)$$

having their counterparts in the components of the turbulent stress tensor

$$\frac{\partial}{\partial x}\overline{(u'^2)}, \quad \frac{\partial}{\partial y}\overline{(u'v')}, \quad \frac{\partial}{\partial z}\overline{(u'w')}, \qquad (4.8)$$

must be subtracted from the right-hand side of (4.2b). These terms can be interpreted as giving rise to "apparent" components which are additive to the convective components

$$\bar{u}(\overline{\partial\theta}/\partial x), \text{ etc.}$$

Hence, for quasi-steady turbulent flow, the equations for the x-component of the mean velocity and the temperature field become

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{\partial}{\partial x}\overline{(u'^2)} - \frac{\partial}{\partial y}\overline{(u'v')} - \frac{\partial}{\partial z}\overline{(u'w')} - -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right), \qquad (4.9a)$$

$$\bar{u}\frac{\partial \bar{\theta}}{\partial x} + \bar{v}\frac{\partial \bar{\theta}}{\partial y} + \bar{w}\frac{\partial \bar{\theta}}{\partial z} = -\frac{\partial}{\partial x}(\overline{u'\theta'}) - \frac{\partial}{\partial y}(\overline{v'\theta'}) - \frac{\partial}{\partial z}(\overline{w'\theta'}) + + a\left(\frac{\partial^2 \bar{\theta}}{\partial x^2} + \frac{\partial^2 \bar{\theta}}{\partial y^2} + \frac{\partial^2 \bar{\theta}}{\partial z^2}\right) + \frac{\nu}{c_p}\Phi. \qquad (4.9b)$$

A further condition of similarity of velocity and temperature fields under these circumstances is that the spectrum and phase of the fluctuation in u' should be similar to that in θ'. This condition cannot be imposed on the flow and it is necessary to establish by observation whether this is actually the case. So far, this has not been done.

The averaging process performed in the Navier–Stokes equations and in the energy equation has not received a precise mathematical justification, and though it appears that the resulting equations are adequate for the analysis of many problems, it must be conceded that they suppress the dependence of the fluctuating components on their spectral characteristics, and contain so many heuristic approximations that they can only be regarded as useful working hypotheses, to be confirmed or refuted for specific circumstances by constant comparison with experiment.

Much effort has been spent measuring velocity fluctuations in isothermal boundary layers, but no systematic measurements have been reported of velocity fluctuations in non-isothermal streams, particularly in streams whose Prandtl number differs appreciably from unity, or of temperature fluctuations in boundary layers, except [13]. In this, Johnson described measurements of velocity and temperature fluctuations in a turbulent boundary layer downstream of a stepwise discontinuity in wall temperature. He found that the (instantaneous) surface of demarkation between unheated and heated fluid was sharp and distinct even within the fully turbulent portion of the momentum boundary layer, so that a probe held in a fixed position within the layer gave intermittent signals of temperature fluctuations. Johnson also found that the local turbulent Prandtl number (Section 9) was not constant across the boundary layer.

Experiments of the type involve a very considerable amount of work, and no measurements extending Johnson's investigations have been reported.

5. BOUNDARY LAYER SIMPLIFICATIONS

The usual boundary layer approximations can be introduced into the principal equations in either of two ways. First, it is possible to start with the averaged equations, and, by estimating terms, to delete those of small order; secondly, it is possible to perform the averaging process in Prandtl's boundary layer equations directly. In both cases, it is impossible to complete a rigorous mathematical derivation owing to the deficiencies in our knowledge of temperature fluctuations. Since the main equations can only be treated as working hypotheses, the same applies *a fortiori* to the averaged boundary layer equations.

Bearing these details in mind, we shall simply insert into the usual two-dimensional boundary layer equations [16] the x-components of the apparent stresses and fluxes, and write:

Conservation of momentum

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{\partial}{\partial y}(\tau_l+\tau_t) \quad (5.1)$$

where

$$\tau_l=\mu\frac{\partial \bar{u}}{\partial y} \quad (5.1a)$$

and

$$\tau_t=-\rho\overline{(u'v')}. \quad (5.1b)$$

In this form the equation takes account of the variation of viscosity with temperature. A more detailed estimation of terms would show that this is true on the assumption that

$$\frac{\Delta\mu}{\mu}\ll\left(\frac{L}{\delta}\right)^2, \quad (5.1c)$$

where L is a characteristic dimension of the body, δ is the boundary layer thickness, and $\Delta\mu$ denotes the maximum difference in the viscosity of the fluid across the boundary layer. In general, this condition will be satisfied.

Conservation of mass

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0. \quad (5.2)$$

Conservation of energy

$$\rho c_p\left(\bar{u}\frac{\partial \bar{\theta}}{\partial x}+\bar{v}\frac{\partial \bar{\theta}}{\partial y}\right)=\frac{\partial}{\partial y}(\dot{q}_l+\dot{q}_t) \quad (5.3)$$

where

$$\dot{q}_l=k\frac{\partial \bar{\theta}}{\partial y} \quad (5.3a)$$

and

$$\dot{q}_t=-\rho c_p\overline{(v'\theta')}. \quad (5.3b)$$

The above form of the energy equation takes into account the dependence of thermal conductivity on temperature.

It is noted that in these, as in the general equations, the Reynolds stresses and fluxes τ_t and $\dot{q}_t$ are additive to the corresponding molecular terms τ_l and $\dot{q}_l$ and that the energy equation is linear.

In the analysis of actual problems, it is not usual to retain the fluctuating terms explicitly in the equation, because no attempts are made to evaluate the fluctuating components and their correlations directly. Instead, following Boussinesq [17], their effect is represented by terms which are given the same appearance as the corresponding viscous and heat conduction terms. Thus, to the molecular viscosity μ there will correspond a coefficient of "eddy" viscosity μ_t, their sum being equal to the "effective" viscosity μ_e. Consequently, the shearing stress can be written

$$\tau=(\mu+\mu_t)\frac{\partial \bar{u}}{\partial y}$$
$$=\mu_e\left(\frac{\partial \bar{u}}{\partial y}\right). \quad (5.4a)$$

In an analogous manner, the heat flux can be written

$$\dot{q}=(k+k_t)\frac{\partial \bar{\theta}}{\partial y}$$
$$=k_e\left(\frac{\partial \bar{\theta}}{\partial y}\right), \quad (5.4b)$$

k_t and k_e denoting the turbulent and effective thermal conductivities, respectively.*

The eddy coefficients, being defined in terms of the oscillating components in the flow and temperature fields, are not properties of the fluid, like μ and k. Their values are generally many magnitudes greater than the molecular transport coefficients. The importance of their use lies in the circumstance that the most effective, heuristic theories of turbulent flow and heat transfer concentrate on making assumptions about them, particularly about the dimensionless ratio

$$Pr_t = \frac{\nu_t}{a_t} = \frac{\mu_t c_p}{k_t} \tag{5.5†}$$

known as the turbulent Prandtl number. We shall also have occasion to use the ratio

$$\epsilon^+ = \frac{\nu_e}{\nu} = \frac{\mu_e}{\mu} = 1 + \frac{\mu_t}{\mu} \tag{5.5a}$$

which has the character of a dimensionless effective viscosity, as well as the effective Prandtl number

$$Pr_e = \frac{\mu_e c_p}{k_e}. \tag{5.5b}$$

In dealing with the energy equation, it is sometimes useful to recall that it can be cast in a more compact form by applying to it the von Mises transformation [18, 19, 16]. In this form, the equation is identical with the one-dimensional Fourier heat-conduction equation in which the conductivity is a function of the space co-ordinate.

6. THE FLOW FIELD

Even though the velocity boundary layer thickness δ is very small, it is necessary to distinguish in it several zones in order to facilitate the analytic description of the flow field. In reality, of course, these zones are not sharply delineated but merge continuously into one another.

(a) The layer immediately adjacent to the wall or the laminar sublayer, now more frequently called the viscous layer. In it, the motion is dominated by the effects of molecular viscosity and $\tau_t \ll \tau_l$ or $\mu_t \ll \mu_l$. The velocity variation is very closely linear with distance $\bar{u} = \mu y$ which implies a constant shearing stress; the extent δ_l of the zone is exceedingly small, varying from $\delta_l = 0{\cdot}02\delta$ to $0{\cdot}000005\delta$ for a range of Reynolds numbers $Re_x = \bar{U}x/\nu = 10^5$ to 10^9. The laminar sublayer is very thin but plays an important part in heat transfer, particularly at high Prandtl numbers and along thermal entry lengths when the thermal boundary layer is also thin, because then the major part of the temperature change takes place in it.

(b) The next layer called the fully turbulent, or constant stress layer is also characterized by a constant shearing stress. The flow is here dominated by turbulent mixing, and

$$\tau_l \ll \tau_t \quad \text{or} \quad \mu \ll \mu_t. \tag{6.1}$$

Extensive experimental evidence indicates that the velocity distribution in this layer is of the universal form

$$u^+ = f(y^+) \tag{6.2}$$

where

$$\left.\begin{aligned} u^+ &= \bar{u}/v_* \\ y^+ &= y v_*/\nu \end{aligned}\right\} \tag{6.3}$$

are the usual dimensionless parameters formed with the friction velocity

$$v_* = \sqrt{\frac{\tau_w}{\rho}}, \tag{6.3a}$$

there being available several empirical expressions for the function f in (6.2). This is the region of Coles' [20] "law of the wall" which extends to a limit variously indicated as

$$\delta_t^+ = 400 \text{ to } 1000$$

and occupies a thickness $\delta_t = \delta$ to $0{\cdot}01\delta$ in the interval of $Re_x = 10^5$ to 10^9 on a flat plate. At the lower Reynolds numbers this region

* The reader may have noticed that the above symbols depart somewhat from those normally encountered in papers on heat transfer where only the quantities ν_t and a_t are give separate designations. The present convention which uses the same symbols for eddy coefficients as those for the corresponding molecular quantities, making distinctions with the aid of suitable subscripts, is more flexible and leads to more symmetrical and typographically simpler expressions.

† This is the inverse of the ratio A_q/A_t favored in the German literature of the subject [16].

nearly fills the boundary layer, except for the laminar sublayer. It is remarkable, and important for heat transfer calculations, that the same expression (6.2) seems to be valid in the presence of pressure gradients and outside rough walls. It cannot be said, either of this layer, or of the laminar sublayer, that the shearing stress remains strictly constant, as is easy to see with reference to the general equation (5.1). Rather, it is asserted that the variation of shearing stress across the narrow zones is small, and can be replaced by a constant value equal to that *at* the wall.

(c) The outermost, wake-like region (also fully turbulent) extends from $y = \delta_t$ to $y = \delta$, and occupies a progressively larger proportion of the boundary layer as it flows along a wall. It is practically non-existent at moderate Reynolds numbers, but at high Reynolds numbers it occupies 0·8 to 0·9 of the extent of the whole turbulent boundary layer. The velocity distribution in this region is sensitive to the pressure gradient and can no longer be described in terms of the form of (6.2), since it must contain a parameter, say $\Pi(x)$, which characterizes the pressure gradient. Coles [21] succeeded in formulating the equation

$$u^+ = A \ln y^+ + B + A\Pi(x)p(y/\delta) \quad (6.4)$$

which contains a new universal function, the "law of the wake", $p(y/\delta)$.

The importance of the wake region in heat transfer calculations has not been examined, and its existence has so far been ignored. Since all present theories of heat transfer confine themselves to the consideration of the laminar sublayer and the fully developed, constant-shear layer, and since the wake region occupies a progressively larger portion of the turbulent boundary layer (but not of a turbulent core in a pipe) as the Reynolds number increases, it would appear that the validity of these theories, when applied to boundary layers (but not to pipes), is restricted to the lower end of the scale of Reynolds numbers, the restriction being more severe in the case of low than in the case of high Prandtl numbers.

(d) The original sharp subdivision into a laminar sublayer and a turbulent layer has proved inadequate in heat transfer studies and it became necessary to distinguish a "transition zone", or a "buffer layer" extending from $y = \delta_l$ to $y = \delta_b$, say, in which the molecular and the eddy coefficients μ and μ_t are of equal importance and comparable magnitude. Experiment indicates that the velocity profile in this region is also described by a universal law of the form of (6.2), being again one independent of pressure gradients, but different in detail.

When the universal parameters u^+ and y^+ are used, it is easy to lose sight of the relative proportions of the various zones to the actual boundary layer thickness. In order to gain some insight, it is useful to derive an approximate equation for the ratios δ_l/δ, δ_b/δ and δ_t/δ in terms of the dimensionless co-ordinates, δ_l^+, δ_b^+ and δ_t^+ and Reynolds number. This can be done by utilizing the available explicit formulae for the boundary layer thickness and for the skin friction coefficient for a flat plate derived from the 1/7th power law. By simple calculation it can then be shown that

$$\frac{y}{\delta} \approx \frac{16}{Re_x^{0\cdot7}} y^+; \quad \left(Re_x = \frac{U_\infty x}{\nu}\right). \quad (6.5)$$

The universal limits $\delta_l^+ = 5$, $\delta_b^+ = 30$, $\delta_t^+ = 1000$, are normally accepted as reasonable values for heat transfer calculations. The ratios δ_l/δ etc. have been expressed in terms of the length Reynolds number Re_x and are seen plotted in the logarithmic diagram of Fig. 4.

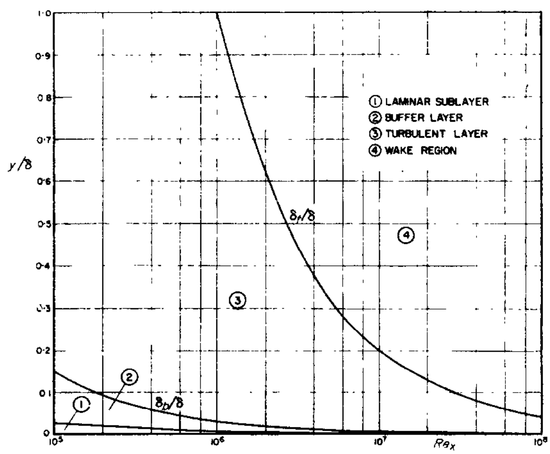

FIG. 4. Extent of the different zones in terms of the length Reynolds number.

The diagram illustrates how the various layers decrease in extent in relation to the boundary layer thickness, and how the wake-like region penetrates further into it with increase of the length Reynolds number.

Table 3. Values of the ratios δ_l/δ, δ_b/δ, and δ_t/δ for various length Reynolds numbers on a flat plate (1/7th power law) which correspond to $\delta^+ = 5$, $\delta^+ = 30$, and $\delta^+ = 1000$

Re_x	Laminar sublayer δ_l/δ	Buffer layer δ_b/δ	Law of the wall δ_t/δ
10^5	0·025	0·152	—
10^6	0·005	0·030	1·000
10^7	0·001	0·006	0·201
10^8	0·0002	0·001	0·040
10^9	0·00004	0·0002	0·008

7. THE LAMINAR SUBLAYER

There seems to be a certain amount of confusion, or controversy, regarding the nature, and even the existence, of the laminar sublayer [22, 23, 24] and it is felt necessary at least to clarify the authors' views. Since this layer is normally inaccessible to direct measurement, much of what is said about it is inevitably conjectural.

For a long time it was held that the velocities in the laminar sublayer do not fluctuate, and that in it laminae of fluid glide over each other, the particles moving steadily along their streamlines in the manner postulated in the mathematical analysis of laminar boundary layers. It was, therefore, found surprising and disturbing that Klebanoff [25] and Laufer [26] were able to detect fluctuations in it. It is true that the root-mean-squares of all three fluctuating components $(\overline{u'^2})^{\frac{1}{2}}$, $(\overline{v'^2})^{\frac{1}{2}}$ and $(\overline{w'^2})^{\frac{1}{2}}$ decay to zero at the wall itself (as they must, owing to the no-slip condition), but the ratio of the root-mean-squares of the longitudinal component u' to the *local* average velocity $\bar{u}$ *increases* across the sublayer and reaches its highest value at the wall. The results of Klebanoff's measurements have been reproduced in Fig. 5; Laufer's measurements

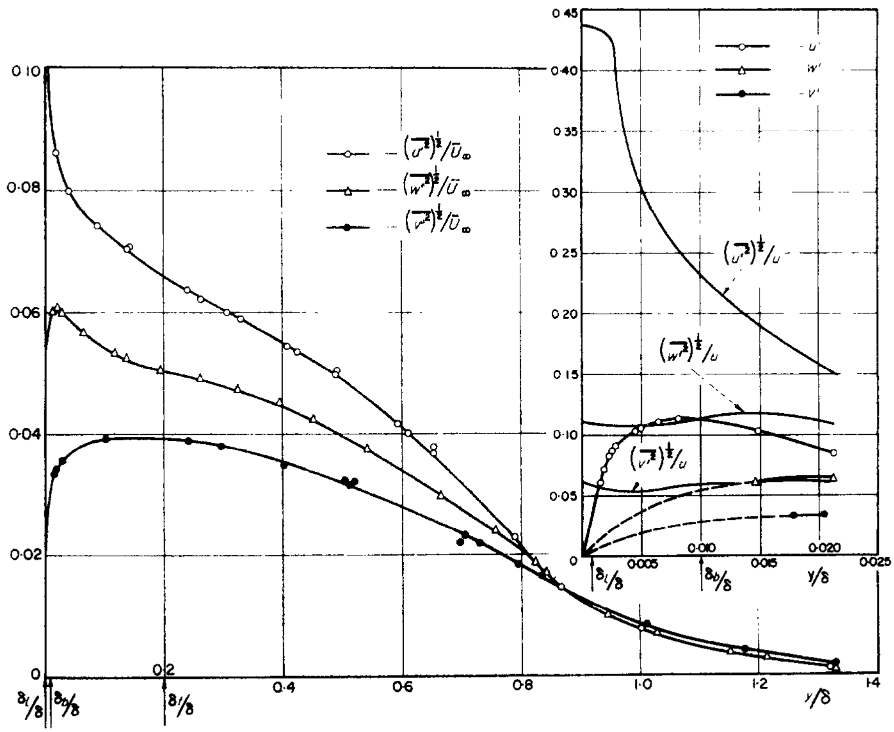

FIG. 5. Measurement of turbulent velocity fluctuations across a turbulent boundary layer on a plate, after Klebanoff [25].

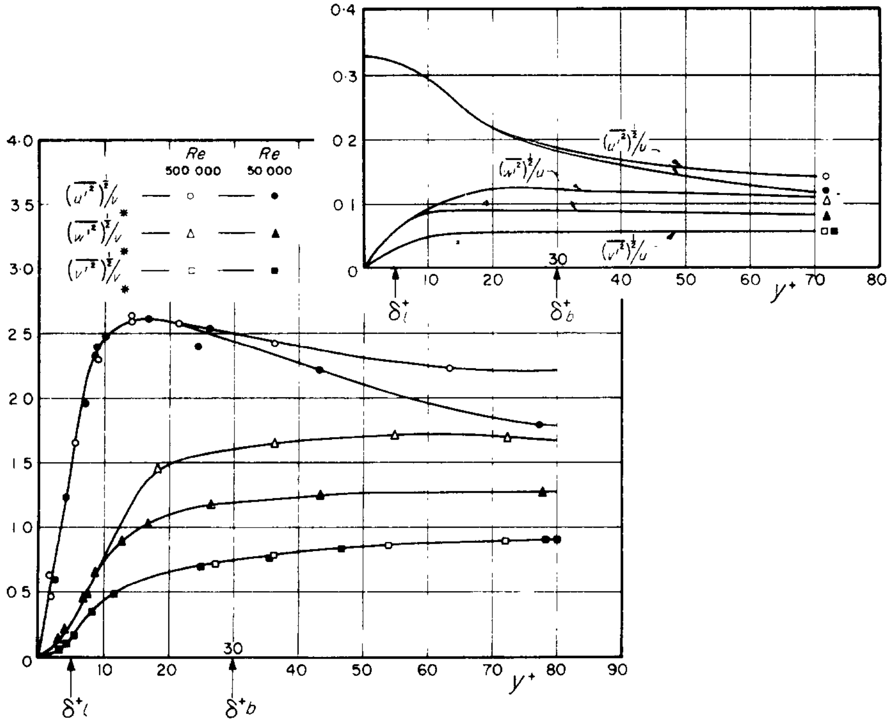

FIG. 6. Measurement of turbulent velocity fluctuations in a pipe, after Laufer [26].

in pipes present an identical pattern, Fig. 6. The measurements are further confirmed when the so-called intermittency factor γ is plotted near the wall, as shown in Fig. 7 which represents the same measurements performed by Klebanoff, as well as the results obtained by Corrsin and Kistler [27].

The preceding results prove conclusively that the laminar sublayer velocity profile is an oscillating one, and make it difficult to reconcile with the usual pattern associated with a steady laminar boundary layer. The authors represent the view that these features admit of an alternative, consistent interpretation.

Recent experimental work on the transfer of heat across laminar boundary layers in the presence of a turbulent free stream performed at Brown University [28, 29, 30, 31, see also 16] indicate that for this (quite normal) condition the velocity profile in a laminar boundary layer carries fluctuations of this type.

At the edge of such a laminar boundary layer the free stream velocity oscillates, as does the velocity adjoining the laminar sublayer of a turbulent flow. For both cases it is to be expected that the velocity profile has a corresponding oscillation. Now, it is well known [16] that such

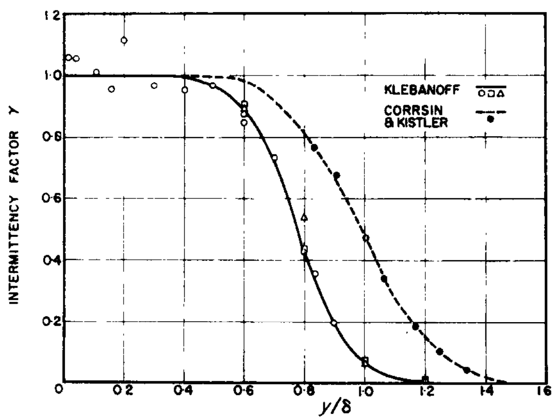

FIG. 7. Variation of intermittency factor γ across a turbulent boundary layer, after Klebanoff [25] and Corrsin and Kistler [27].

oscillations in the free stream are transmitted across a laminar layer by the action of shear, and that the relative amplitude of a harmonic oscillation, for example, increases in the direction of the wall, in qualitative agreement with the results in Figs. 5, 6 and 7. In a laminar boundary layer such disturbances can become amplified, as is proved in the Tollmien–Schlichting theory of stability, and lead eventually to transition.

By its very existence, it is clear that the laminar sublayer is stable though unsteady. The velocity profile in it is very nearly linear, and it can sustain the random disturbances imposed upon it by the adjacent turbulent layers without permitting them to amplify. An oscillating laminar layer or sublayer is distinguished from a turbulent layer in that the latter carries eddies resulting from upstream instabilities which have promoted transition. The magnitude of the resulting motions makes turbulent exchange the dominating mechanism for momentum transfer in a turbulent stream. By contrast, the dominating mechanism in a laminar layer or sublayer is the propagation of shear by molecular viscosity, irrespective of whether this is associated with a steady or an oscillating velocity profile.

The longitudinal oscillations may even give rise to a secondary flow, and this may be responsible for the curious transverse motions observed by Fage and Townsend [32] which prompted Dankwerts [22] and Hanratty [24] to formulate the theory of a detaching and re-attaching sublayer.

If these interpretations are accepted, the contradiction disappears, and the new term "viscous layer" coined for such an oscillating laminar sublayer appears confusing and superfluous. Incidentally, in the light of these remarks, the appropriateness of using the intermittency factor γ as an indicator of the flow regime may be questioned.

Sternberg [33] has recently discussed the sublayer from a complementary viewpoint. On assuming that the mean flow in the sublayer and the turbulent field in the neighboring part of the boundary layer are known from experiment, Sternberg showed that the turbulent velocity fluctuations are directly dissipated by viscosity in the sublayer, and that the production of turbulent energy attains a maximum in a region where the laminar and turbulent shearing stresses are equal.

8. THE LAW OF THE WALL

In addition to Prandtl's and Taylor's semi-empirical expression for the law of the wall, namely

$$u^+ = A \ln y^+ + B, \tag{8.1}$$

commonly employed to describe the universal velocity profile in the turbulent layer, it was felt necessary to develop alternative formulations for further use in heat transfer calculations. In these formulations, which will now be reviewed, it is admitted that the corresponding equation for the laminar sublayer should be

$$u^+ = y^+ \tag{8.2}$$

and the main motivation for the alternative formulae was the desire to provide a smooth rather than an abrupt transition from one expression to the other, in conformity with experimental results, but also with a view to simplifying the ensuing derivations.

Thus, for example, von Kármán [34] introduced a distinct buffer layer, postulating in it a relation of the form of (8.1) with, however, different values for the empirical coefficients A and B. A representative list of formulae proposed by different authors is given in Table 4. The diagram in Fig. 8 shows a comparison between experimental data and equations (8.1) with (8.2). The values for the constants are those recommended by Coles [20], namely

$$A = 2{\cdot}5; \quad B = 5{\cdot}1. \tag{8.3}$$

The diagram clearly shows the need for smoothing out the transition between these two equations.

An expression which encompasses the law of the wall in one equation was suggested by van Driest [35].* Its principal interest lies in the fact that good agreement with experiment has been achieved with the aid of considerations regarding the laminar sublayer which are essentially identical with those advanced in Section 7. It arises by the application of a

* The first equation of this type is due to Reichardt [43].

Table 4. Various analytical forms proposed for the universal velocity profile

Formula for $u^+(y^+)$	$\epsilon^+ = dy^+/du^+$	Range	Author and reference
$u^+ = y^+$ $u^+ = 2{\cdot}5 \ln y^+ + 5{\cdot}5$	$\epsilon^+ = 1$ $\epsilon^+ = 0{\cdot}4y^+$	$0 \leqslant y^+ \leqslant 11{\cdot}5$ $y^+ > 11{\cdot}5$	Prandtl and Taylor [16]
$u^+ = y^+$ $u^+ = 5 \ln y^+ - 3{\cdot}05$ $u^+ = 2{\cdot}5 \ln y^+ + 5{\cdot}5$	$\epsilon^+ = 1$ $\epsilon^+ = 0{\cdot}2y^+$ $\epsilon^+ = 0{\cdot}4y^+$	$0 \leqslant y^+ \leqslant 5$ $5 \leqslant y^+ \leqslant 30$ $y^+ > 30$	von Kármán [34]
$u^+ = 14{\cdot}53 \tanh (y^+/14{\cdot}53)$ $u^+ = 2{\cdot}5 \ln y^+ + 5{\cdot}5$	$\epsilon^+ = 1 + \sinh^2 (y^+/14{\cdot}53)$ $\epsilon^+ = 0{\cdot}4y^+$	$0 \leqslant y^+ \leqslant 27{\cdot}5$ $y^+ > 27{\cdot}5$	Rannie [44]
$\frac{dy^+}{du^+} = \frac{2}{1 + \{1 + 4K^2y^{+2}[1 - \exp(-y^+/A^+)]^2\}^{1/2}}$ $K = 0{\cdot}4 \quad A^+ = 26$	$\epsilon^+ = 1 + \{1 + 4K^2y^{+2}[1 - \exp(-y^+/A^+)]^2\}^{1/2}$	all y^+	van Driest [35]
$u^+ = 2{\cdot}5 \ln (1 + 0{\cdot}4y^+)$ $+ 7{\cdot}8[1 - \exp(-y^+/11)$ $- (y^+/11) \exp(-0{\cdot}33y^+)]$	$\frac{1}{\epsilon^+} = \frac{1}{1 + 0{\cdot}4y^+} + 7{\cdot}8[(1/11) \exp(-y^+/11)$ $- (1/11) \exp(-0{\cdot}33y^+) + 0{\cdot}03y^+ \exp(-0{\cdot}33y^+)]$	all y^+	Reichardt [43]
$\frac{du^+}{dy^+} = \frac{1}{1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]}$ $n = 0{\cdot}124$ $u^+ = 2{\cdot}78 \ln y^+ + 3{\cdot}8$	$\epsilon^+ = 1 + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]$ $\epsilon^+ = y/2{\cdot}78$	$0 \leqslant y^+ \leqslant 26$	Deissler [36]
$y^+ = u^+ + A[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2$ $- \frac{1}{6}(Bu^+)^3 - \frac{1}{24}(Bu^+)^4]$ (last term in u^{+4} may be omitted)	$\epsilon^+ = 1 + AB[\exp Bu^+ - 1 - Bu^+ - \frac{1}{2}(Bu^+)^2$ $- \frac{1}{6}(Bu^+)^3]$ (last term in u^{+3} may be omitted)	all y^+ $A = 0{\cdot}1108$ $B = 0{\cdot}4$	Spalding [45]

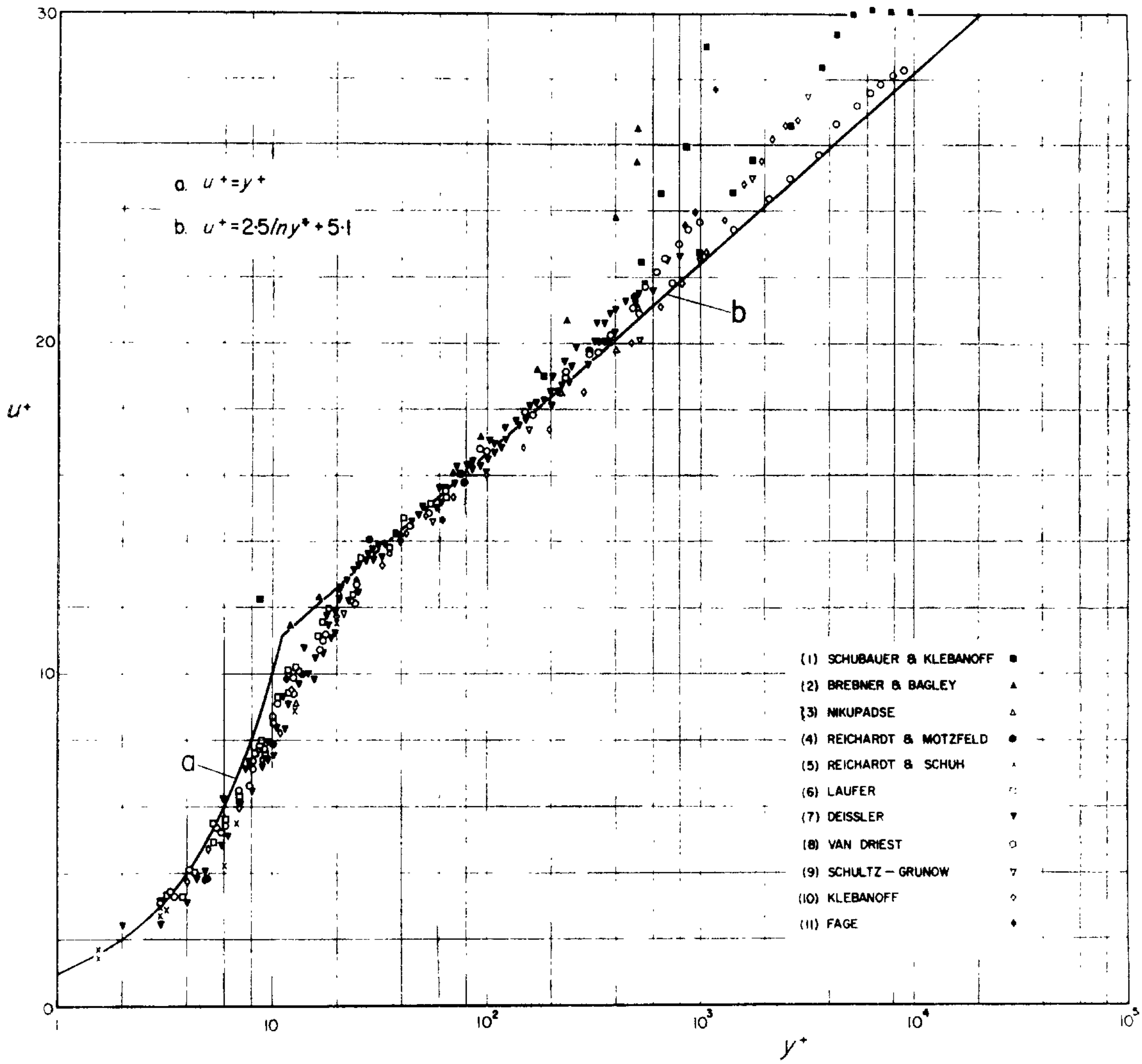

FIG. 8. Comparison of the law of the wall with experiment [15, 16, 25, 35–42].

damping factor, $[1 - \exp(-y/A)]$ where A is a constant, to the term $\overline{u'v'}$, the damping factor being "borrowed" from Stokes' well-known solution for the viscous decay of oscillations above a flat plate. These assumptions lead to the following expression for the shearing stress

$$\tau = \mu\left(\frac{\partial \bar{u}}{\partial y}\right) + \rho K^2 y^2 [1 - \exp(-y/A)]^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2, \tag{8.4}$$

where K is another constant. This relation can be reduced to the universal form

$$\frac{\tau}{\tau_w} = \frac{\partial u^+}{\partial y^+} + K^2 y^{+2}[1 - \exp(-y^+/A^+)]^2 \left(\frac{\partial u^+}{\partial y^+}\right)^2, \tag{8.5}$$

which is equivalent to the statement that the eddy viscosity is given by

$$\epsilon^+ = 1 + \frac{\mu_t}{\mu} = \frac{\tau}{\tau_w} \cdot \frac{\mathrm{d}y^+}{\mathrm{d}u^+}. \tag{8.6}$$

Here the sign of ordinary differentiation can be used, since u^+ is uniquely related to y^+.

The further assumption that

$$\tau \approx \tau_w \tag{8.7}$$

leads to van Driest's law of the wall. Strictly speaking, the assumption (8.7) implies a number of restrictions which will be examined in greater detail in Section 12. It is, however, noteworthy that the resulting law of the wall seems to reproduce experimental results obtained under conditions when the former are not satisfied, and it might be conjectured that the corresponding expression

$$\epsilon^+ = \frac{dy^+}{du^+} \tag{8.8}$$

enjoys wider validity than is implied in this derivation.

The most recently proposed expression, due to Spalding [45], also succeeds in providing a single analytically smooth expression for all three layers. This has been achieved by inverting the relation and by writing it in the form $y^+(u^+)$. Since the equation was proposed very recently, the best values for the coefficients A and B are somewhat in doubt. The form of the the equation shows that $y^+ \to u^+$ for $u^+ \to 0$ and that near the wall the effective, dimensionless kinematic viscosity, (8.6),

$$\epsilon^+ = 1 + AB[e^{Bu^+} - 1 - Bu^+ - \tfrac{1}{2}(Bu^+)^2] \tag{8.9}$$

increases with y^+ as $1 + (y^+)^3$, in agreement with Reichardt [46, 47].* The equivalent expression gives ϵ^+ increasing as $1 + (y^+)^4$.

The diagram in Fig. 9 shows a comparison between the different laws of the wall listed in Table 4 and reveals that the discrepancies between them are of the same order of magnitude as the scatter in the experimental data in Fig. 8. There seem to be, therefore, no physical grounds for discriminating between the various proposals and the final choice can be made on grounds of convenience. This undoubtedly favours Spalding's formulation. It is sometimes stated that more precise measurements would remedy this deficiency, but the alternative view that a universal law is only approximate cannot be dismissed. It is plausible that minor, unaccounted influences are responsible for these deviations.

For our purposes, it is necessary to record here that the universal law of the wall, in any of the forms listed in Table 4, seems to describe the conditions in the fully developed layer adequately, and irrespectively of the value of the pressure gradient. It can even be extended to compressible flows [51, p. 546]. Nevertheless, it is important to bear in mind that its validity has been established essentially in connection with isothermal streams, and thus it is not known positively whether a highly variable viscosity in the presence of a thermal field would affect the law of the wall.

The logarithmic law of the wall, together with the proportionality relation $u^+ = y^+$ for the laminar sublayer, or the alternative analytic formulations listed in Table 4, do not lead to a direct, explicit representation of the average velocity profile $\bar{u}$ in terms of the co-ordinates x and y. They should be regarded as formulations which remove the need for making explicit assumptions for the eddy or effective viscosity in (5.4). The actual velocity profiles, and the shearing stress distribution for any given body shape, must be obtained by an integration of (5.1). This, however, has not yet been performed, but limited success was achieved by the use of its integral form

$$\frac{d\delta_2}{dx} + (H + 2)\frac{\delta_2}{\bar{U}}\frac{d\bar{U}}{dx} = \frac{\tau_w}{\rho \bar{U}^2} \tag{8.10}$$

where δ_2 is the momentum thickness, $\bar{U}(x)$ is the free-stream velocity, and τ_w is the shearing stress at the wall. The form parameter H is defined as

$$H = \frac{\delta_1}{\delta_2} \tag{8.11}$$

where δ_1 is the displacement thickness. Even this equation has been solved for a limited number of cases only, among which solutions for those forms of the law of the wall which are most useful in heat transfer are not included. This is a task which still remains to be done, particularly in relation to Spalding's equation (8.10).

There is one final aspect of the flow in a

* See also Townsend [48] p. 220, Elrod, Jr. [49], and Hinze [50].

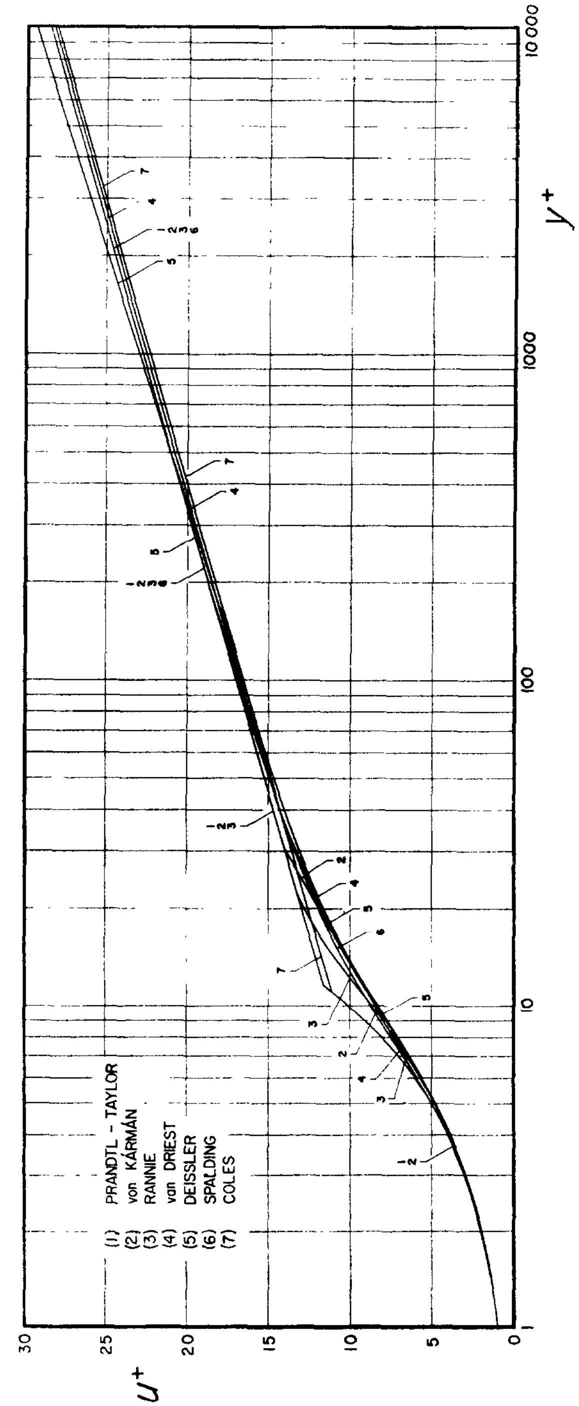

FIG. 9. Various analytic forms proposed for the universal velocity profile.

turbulent boundary layer which is worthy of attention. Since the mean flow in the boundary layer satisfies the continuity equation (4.1a), it is interesting to consider the stream function ($u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$) which is given by $\psi = \int_0^y u \, dy$, the integration being performed at $x =$ constant. Introducing the similarity variables u^+ and y^+ from (6.3) and (6.3a), it is easy to see that $\psi = \nu \int_0^{y^+} u^+ \, dy^+$. Thus it follows that in the laminar sublayer and in the fully turbulent layer, including the buffer layer, the stream function ψ is uniquely related to y^+, and so, in view of the similarity relation (6.2) also to u^+; in other words, lines of constant u^+ and ψ are identical, and there exists a relation $\psi(u^+) = \int_0^{u^+} u^+ \, (dy^+/du^+) \, du^+$.

This fact implies that the rate of flow through the laminar sublayer which extends from $y^+ = 0$, $(u^+ = 0)$ to $y^+ = 5$, $(u^+ = 5)$ is constant. The same would appear to be true regarding the buffer layer ($5 \leqslant y^+ \leqslant 30$), but not of the fully developed turbulent layer whose edge does not coincide with a fixed value of y^+. Hence, the laminar sublayer, apart from appearing to be stable with respect to disturbances, also passes a fixed volume or mass rate of flow ($\rho =$ constant), presumably determined during the process of transition. The volume rate of flow through the sublayer is then $\psi = 12{\cdot}5\nu$ per unit width, and the rate of flow through the buffer layer is about $\psi = 440\nu$.

9. EDDY CONDUCTIVITY AND TURBULENT PRANDTL NUMBER

In view of the difficulties attendant upon the solution of the equation of motion (5.1) it might appear that any progress with the energy equation (5.3) or (5.6) will be impeded by the lack of knowledge of the velocity profile $u(x, y)$. This, however, is not the case, and the theory of heat transfer can nevertheless be pursued, because sufficient information is available in the universal law of the wall. The only quantity which remains to be discussed, before this final problem can be attacked, is the eddy conductivity.

The simplest approach is to make a plausible assumption, and this usually takes the form of an assumption concerning the turbulent Prandtl number Pr_t. The first statement of this type was made by Reynolds [3] who reached the conclusion that

$$Pr_t = 1 \tag{9.1}$$

on the basis of a heuristic argument during which he noted that in a fully turbulent field, both momentum and heat are transferred as a result of the motion of eddies. Details of this argument can be found in [42, 52]. The soundness of the preceding, exceedingly simple assumption has been questioned by many research workers, and numerous attempts have been made to obtain direct experimental evidence about it [8, 12, 53, 54]. At the present time, no unified and consistent picture emerges. In particular, it is not clear whether the turbulent Prandtl number is completely independent of the molecular Prandtl number, as implied in assumption (9.1) and in many other theories. We now propose to give a brief account of these conflicting investigations, beginning with that due to Ludwieg [53], because it bears an aura of credibility, and provides a link to Taylor's [55, 16] vorticity transport theory. Measurements presently available are for turbulent heat transfer in gases, for which the molecular Prandtl number is close to unity. Whilst gases are generally more convenient experimentally than liquids, their use necessitates care if the small variations of apparent Prandtl number are to be clearly distinguishable from experimental uncertainties. This is especially important when evaluation of results involves differentiation of experimentally determined profiles.

Ludwieg [53] measured the variation with radius of the turbulent Prandtl number for air flowing in a pipe, as illustrated in Fig. 10. These measurements indicate that the turbulent Prandtl number varies smoothly and continuously from a value of about 0·7 near the pipe wall to a value approaching 0·5 at the centre of the pipe. Flow at the centre of a pipe does not include regions of wake flow, such as occur in boundary layers beyond the law-of-the-wall region. However, an extrapolation of Ludwieg's results on the basis of the reciprocal of distance from the wall shows that his measured values are asymptotic to a turbulent Prandtl number of 0·5 at large distances from the wall, where the

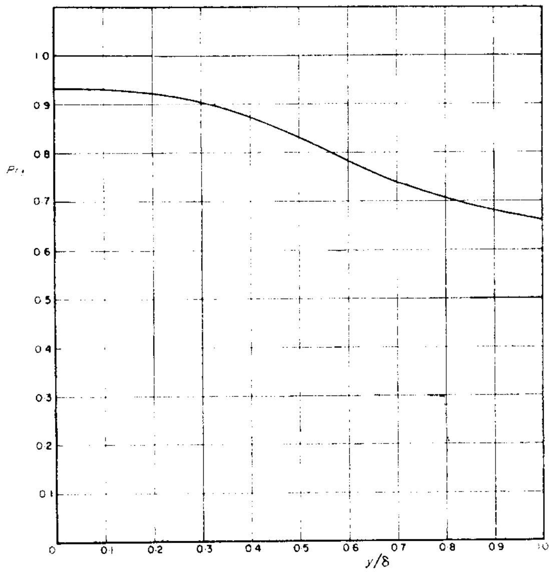

FIG. 10. Variation of turbulent Prandtl number Pr_t with distance from wall, after Ludwieg [53].

law of the wake would apply. This is in agreement with the value of 0·5 measured in the wake of a cylinder by Fage and Faulkner [56, 16], and by Reichardt in a free jet [43]. The value of 0·5 is also indicated by Taylor's vorticity transport theory, and this gives support to the correctness of the values measured, and to the trend with position relative to the wall of Ludwieg's results.

Unfortunately the results of the remaining measurements do not agree with those of Ludwieg, nor between themselves. These results, together with those of Ludwieg, are shown in Fig. 11. From the diagram it can be seen that most measurements have trends opposite to Ludwieg's, and there is great disagreement between them. It may be remarked that the uncertainties involved in the experiments are difficult to overcome, and it has been pointed out [57] that there is a distinct lack of agreement between local measurements and overall surface-flux measurements.

There have been several attempts to provide analyses of the variation of turbulent Prandtl number, some proposing that the ratio of the eddy diffusivities must be unity, e.g. [58] who reinforced the original argument due to Reynolds by one based on consideration of the Lagrangian description of eddy motion. Others [59, 60, 61] have suggested various forms of variation; that due to Jenkins [59] has been compared with experiment but no distinct confirmation of any analysis appears to have been found. It is therefore clear that the question of the turbulent Prandtl number is still wide open, and merits investigation not only for air but also for fluids of a wide range of molecular Prandtl number. In this paper we shall assume that Ludwieg's results are closest to being correct, for the (admittedly insufficient) reason that they correspond best with gross characteristics as already discussed. It might be pertinent to remark that all existing theories of heat transfer either assume $Pr_t = 1$ or an average, but constant value, say $Pr_t = 0{\cdot}78$. This has not prevented them from giving results which are acceptably close for most purposes.

10. THE REYNOLDS ANALOGY

Before proceeding with an account of the established relations for the rates of heat transfer in turbulent flow it is useful to derive a general relation between the shearing stress at the wall, τ_w, and the heat flux at the wall, $\dot{q}_w$, known under the name of the Reynolds analogy. It was first derived by Reynolds [62, 42, 52] by applying a somewhat different line of reasoning from the one given here. In this section we shall show that the analogy is a direct consequence of the general principle of similarity for the Navier–Stokes equations, discussed in Section 4. In the case of a flat plate at zero incidence and for fluids whose Prandtl number is equal to unity, the equations for the u-component of velocity and for the temperature difference θ become similar, provided that u and θ obey similar boundary conditions in x, y, and time. In the case of a uniform mean wall temperature $\bar{T}_w$, of uniform mean velocity $\bar{U}_\infty$, and of a uniform mean temperature difference θ_∞, complete similarity can exist if, in addition,

$$\partial u/\partial t = \partial u'/\partial t \quad \text{and} \quad \partial \theta/\partial t = \partial \theta'/\partial t \qquad (10.1)$$

are similar. This latter condition is plausible for a boundary layer, but, of course, it is in need of direct experimental verification. If this is the case, then the mean, normalized profiles

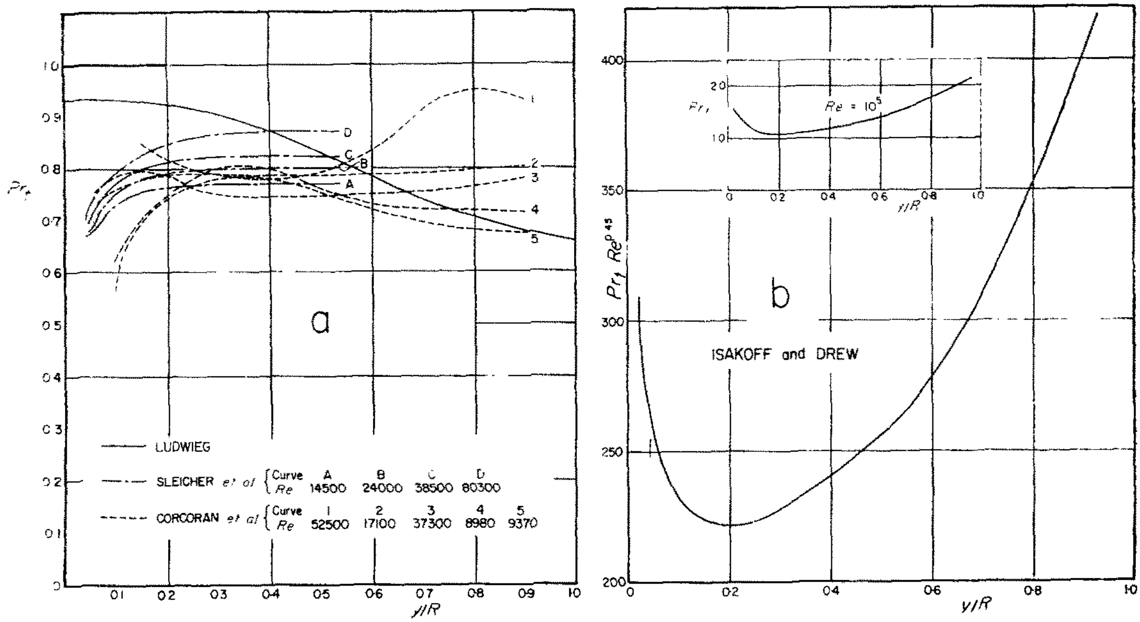

FIG. 11. Variation of turbulent Prandtl number Pr_t with distance from wall. Comparison of different results.

(a) Ludwieg [53] ———— Corcoran *et al.* [8] - - - -

Sleicher [54] — · — · — ·

A. $Re = 14500$
B. $Re = 24000$
C. $Re = 38500$
D. $Re = 80300$

1. $Re = 52500$
2. $Re = 17100$
3. $Re = 37300$
4. $Re = 8980$
5. $Re = 9370$

(b) Isakoff and Drew [12].

$\bar{u}/\bar{U}_w$ and $\bar{\theta}/\bar{\theta}_\infty$ must become identical functions of the transverse co-ordinate y at every cross section x, since both satisfy the boundary conditions

$$\frac{\bar{u}}{\bar{U}_\infty} = \frac{\bar{\theta}}{\bar{\theta}_\infty} = 1 \text{ at } y = \infty$$

$$\frac{\bar{u}}{\bar{U}_\infty} = \frac{\bar{\theta}}{\bar{\theta}_\infty} = 0 \text{ at } y = 0.$$

Since

$$\tau_w = \mu\bar{U}_\infty \cdot \left[\frac{\partial(\bar{u}/\bar{U}_\infty)}{\partial y}\right]_w \tag{10.2}$$

and

$$\dot{q}_w = k\bar{\theta}_\infty \left[\frac{\partial(\bar{\theta}/\bar{\theta}_\infty)}{\partial y}\right]_w, \tag{10.2a}$$

it follows that

$$\frac{\tau_w}{\mu\bar{U}_\infty} = \frac{\dot{q}_w}{k\bar{\theta}_\infty} \quad \text{(all } x\text{)} \tag{10.3}$$

which is a form of the Reynolds analogy. It is usual to re-arrange this relation by introducing the local skin friction coefficient

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}_\infty^2} \tag{10.4}$$

and the local Stanton number

$$St = \frac{Nu}{RePr} = \frac{\dot{q}_w}{\rho c_p \bar{U}_\infty \bar{\theta}_\infty}, \tag{10.5}$$

when (10.3) assumes the very simple form

$$St = \tfrac{1}{2}c_f. \tag{10.6}$$

The strict validity of (10.6) is seen to be very limited, but it appears that it applies, approximately, for other than the above sets of conditions. Its appeal lies in its simplicity and its recurring utility in circumstances where it does strictly apply, so that it has developed something of a reputation as a panacea, and attempts have been made to extend it by the application of various forms of plausible reasoning [63, 64, 65, 42, 34, 57, 44, 66]. On occasion it has been used rather recklessly, and applied under conditions when it could not possibly be correct. In this connection the survey paper by Sherwood [67] is worthy of mention.

11. MATHEMATICAL FORMULATION OF PROBLEM

In principle, and subject to the various reservations and simplifications expressed earlier, the temperature field is determined by the energy equation (5.3) and the boundary conditions of each problem. The velocity components u and v are implied in the equation of motion (5.1), in which the (experimental) universal law of the wall provides the expression for the eddy viscosity μ_t in (5.4), and the eddy conductivity, k_t, in (5.4a), is in turn determined by the turbulent Prandtl number, Pr_t, discussed in Section 10. Thus all the information required for the integration of the energy equation is available, if incompletely. Consequently, within the limitations outlined earlier, the problem of heat transfer across turbulent layers has been reduced to mathematical terms. The method of solving it in those terms has been provided by Spalding [1], and will be discussed in Section 14.

The present practice in heat transfer calculations is still confined to the more elementary methods and it is necessary to precede our account of the exact theory with a review of the procedures in which some of the mathematical steps outlined above are replaced by hypotheses of a physical nature.

12. ELEMENTARY THEORIES OF TURBULENT CONVECTION

All elementary theories of turbulent convection take as their starting point the Boussinesq expressions for shear stress, equation (5.4) and for heat flux, equation (5.4a) and concentrate their efforts on the consideration of the ratio $\tau/\dot{q}$, in a manner suggested by the Reynolds analogy (10.3). The two quantities are linked analytically in that the thermal conductivity k as well as the eddy conductivity k_t are expressed in terms of the appropriate Prandtl numbers. Thus (5.4) and (5.4a) can be re-written as follows:

$$\tau = (\mu + \mu_t)\frac{\partial \bar{u}}{\partial y} \qquad (12.1)$$

$$\dot{q} = c_p\left(\frac{\nu}{Pr} + \frac{\nu_t}{Pr_t}\right)\frac{\partial \bar{\theta}}{\partial y}. \qquad (12.2)$$

Instead of substituting these expressions into the equations of energy and motion, as outlined in the preceding section, the assumption is made that the ratio $\tau/\dot{q}$ at any given value of x remains constant in the transverse direction y. This hypothesis removes the need to make any further reference to the partial differential equations, and it becomes necessary to investigate the implications of such a sweeping assumption.

Referring to (5.1) and (5.3), it is apparent that the shearing stress τ and the heat flux $\dot{q}$ can be expressed in the form of integrals with respect to y:

$$\tau - \tau_w = \int_0^y \left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial u}{\partial y} - \bar{U}\frac{\mathrm{d}\bar{U}}{\mathrm{d}x}\right)\mathrm{d}y \qquad (12.3)$$

$$\dot{q} - \dot{q}_w = c_p\int_0^y \left(\bar{u}\frac{\partial \bar{\theta}}{\partial x} + \bar{v}\frac{\partial \bar{\theta}}{\partial y}\right)\mathrm{d}y. \qquad (12.4)$$

The basic assumption also implies that

$$\frac{\tau}{\dot{q}} = \frac{\tau_w}{\dot{q}_w} = \text{constant} \quad \text{(at } x = \text{constant and any } y) \qquad (12.5)$$

so that

$$\frac{\tau - \tau_w}{\dot{q} - \dot{q}_w} = \text{constant},$$

and constancy of this ratio would demand that the two functions of x and y resulting from the integrations on the right-hand sides of (12.3) and (12.4) must at most differ by a constant factor of proportionality. This would occur if the temperature and velocity fields $\bar{\theta}(x, y)$ and $\bar{u}(x, y)$ became similar and if no pressure gradient were present. Excluding very special combinations of parameters, this would occur under the same restrictions for which the Reynolds analogy remains valid, Section 10. It follows that the ratio $\tau/\dot{q}$ cannot remain constant across the boundary layer exactly except in very special circumstances. Nevertheless, the preceding assumption has led to very useful formulae, and this suggests that it is satisfied with a sufficient degree of accuracy. Such conditions exist when the rate of variation in velocity and temperature in the flow direction is several orders of magnitude smaller than that in the transverse direction. Consequently, the theories based on assumption (12.5) are likely to succeed for fully developed flows in pipes or channels, or for boundary layer flows with small pressure gradients.

Under such conditions

$$\frac{\partial \bar{u}}{\partial x} \approx 0 \quad \text{hence} \quad \frac{\partial \bar{v}}{\partial y} \approx 0 \quad \text{and} \quad \bar{v} \approx 0 \qquad (12.5a)$$

so that with

$$\frac{d\bar{p}}{dx} \approx 0 \qquad (12.5b)$$

the integrands in (12.3) and (12.4) become small, can be replaced by average values, and the ratio $\tau/\dot{q}$ becomes nearly independent of y. In particular, when the ratio of $\tau_w/\dot{q}_w$ is examined, it is convenient to consider their values in terms of the energy integral equation (2.3) and the usual momentum integral equations

$$\frac{\partial}{\partial x}\int_0^{\Delta} \bar{u}(\bar{U} - \bar{u})\,dy + \frac{d\bar{U}}{dx}\int_0^{\Delta} (\bar{U} - \bar{u})\,dy = \frac{\tau_w}{\rho}, \qquad (12.6)$$

in order to realize that $\tau/\dot{q}_w$ can remain strictly constant, and independent of x, only under very special circumstances.

It is clear that a theory based on this assumption will lead to an expression of the heat flux $\dot{q}_w$ at the wall in terms of the shearing stress τ_w, and can, therefore, be described as an "analogy". Often, theories of this class are described as extensions of the Reynolds analogy, because the variation of $\dot{q}_w$ with x becomes "analogous" to that of τ_w, as was postulated in the first place, (12.5).

It can be verified immediately that for $Pr = Pr_t = 1$, assumption (12.5) leads directly to the conclusion that

$$\frac{\partial \bar{u}/\partial y}{\partial \bar{\theta}/\partial y} = \text{constant}, \quad (\text{at any } y \text{ for } x = \text{constant})$$

as seen from (12.1) and (12.2). Thus, the simple theory will be consistent with the existence of Reynolds' analogy in this special case. A comparison with the argument of Section 10 suggests, therefore, that assuming $Pr_t = 1$ is equivalent to the statement that the temperature fluctuations θ' are spectrally similar to the velocity fluctuations u'. As mentioned earlier, no experimental evidence regarding this statement is at present available.

On comparing (12.3) with the conditions (12.5a) and (12.5b) it is apparent that they also lead to the statement that $\tau \approx \tau_w$ which was mentioned earlier in Section 8, equation (8.7). It will be recalled that it was necessary to assume virtual constancy of the shearing stress in order to derive (8.9) which plays an important part in the exact theory to be described in Section 14.

In the early extensions of Reynolds' analogy, such as those due to Prandtl [4, 68] or Taylor [5], the temperature field is determined by integration, starting with*

$$\frac{d\bar{\theta}}{dy} = \left(\frac{\dot{q}_w}{\tau_w}\right)\frac{\nu + \nu_t}{\nu/Pr + \nu_t/Pr_t}\cdot\frac{1}{c_p}\frac{d\bar{u}}{dy}. \qquad (12.7)$$

In order to make this possible, it is stipulated, quite arbitrarily, that the boundary layer can be divided sharply into a laminar sublayer and a fully developed turbulent layer, without direct reference to the actual conditions prevailing in them. It is assumed that in the laminar sublayer, extending from the wall to a provisional distance y_l, the eddy coefficients are negligible. Hence

$$\frac{d\bar{\theta}}{dy} = \left(\frac{\dot{q}_w}{\tau_w}\right)\frac{Pr}{c_p}\frac{d\bar{u}}{dy},$$

or, by integration

$$\bar{\theta}_l = \left(\frac{\dot{q}_w}{\tau_w}\right)\frac{Pr}{c_p}\,\bar{u}_l \qquad (12.8)$$

where $\bar{\theta}_l$ and $\bar{u}_l$ are the values of temperature difference and velocity at y_l.

The second integration is performed from the edge of the boundary layer inwards, and it is

* It is now possible to replace the signs of partial differentiation by those of total differentiation, because integration is performed exclusively at x = constant and with respect to y only.

assumed that in the turbulent core the molecular coefficients are negligible. In addition, it is assumed that $Pr_t = 1$, but the assumption of a constant value, as discussed in Section 9, would still permit the method to be applied. Hence

$$\frac{d\bar{\theta}}{dy} = \left(\frac{\dot{q}_w}{\tau_w}\right)\frac{1}{c_p}\frac{d\bar{u}}{dy}$$

or

$$\bar{\theta}_\infty - \bar{\theta} = \left(\frac{\dot{q}_w}{\tau_w}\right)\frac{1}{c_p}(\bar{U}_\infty - \bar{u}_l). \qquad (12.9)$$

Elimination of the constant ratio $(\dot{q}_w/\tau_w)$ from (12.8) and (12.9) leads to a relation between $\bar{\theta}_l$ and $\bar{u}_l$ in the form

$$\frac{\bar{\theta}_\infty}{\bar{\theta}_l} = 1 + \frac{1}{Pr}\left(\frac{U_\infty}{\bar{u}_l} - 1\right) \qquad (12.10)$$

which can be used, in turn, to determine the constant ratio $\dot{q}_w/\tau_w$, when we obtain

$$\dot{q}_w = \frac{c_p \bar{\theta}_\infty / \bar{U}_\infty}{1 + (\bar{u}_l/\bar{U}_\infty)(Pr - 1)}\tau_w$$

(Prandtl–Taylor). (12.11)

This constitutes the fundamental result of the theory; it was derived independently by Prandtl and Taylor. In order to complete the derivation, it is necessary to stipulate a value for $\bar{u}$ at the arbitrary boundary separating the two layers. This can be done directly with reference to the universal velocity profile, or indirectly, by fitting (12.11) to experimental results in heat transfer.*

From the preceding derivation it is clear that the theory implies two serious limitations on its applicability to Prandtl numbers whose values differ appreciably from unity. At $Pr = 1$, the relation merely reduces to the Reynolds analogy once more. At very low Prandtl numbers, the derivation is in serious error, because then the thermal boundary layer is much larger than the velocity boundary layer, Fig. 12. This means that the limits of integration in (12.9) have been chosen improperly, because the temperature

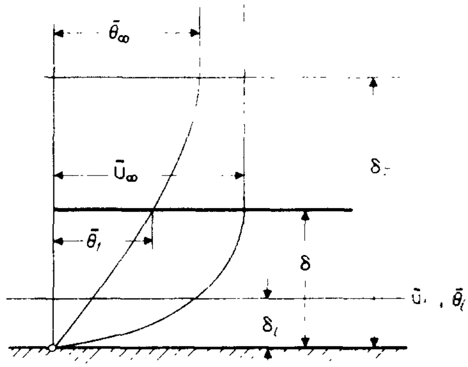

FIG. 12. Very low Prandtl number.

difference at the edge of the velocity boundary layer, $\bar{\theta}_t$, differs considerably from $\bar{\theta}_\infty$. Furthermore, the omission of the term ν/Pr from (12.2) is no longer justified, because the term can become comparable with ν_t/Pr_t, even in the turbulent zone, when $Pr \ll 1$, as is the case with molten metals.

At very high Prandtl numbers, the opposite relation between the two boundary layer thicknesses prevails, and the assumptions are justified. Thus the method is valid for moderately high or very high Prandtl numbers. In the latter case the arbitrary subdivision of the boundary layer into two sharply delineated zones constitutes a serious limitation and must eventually be replaced by one which more nearly fits the actual circumstances.

Since the preceding elementary theory is still widely used in practice we shall record the final result which is obtained when the explicit expression for shearing stress obtained from the 1/7th power law is substituted in it, and when a value $\bar{u}_l$ is chosen which corresponds to $\delta_l^+ = 5$. Then

$$\frac{\bar{u}_l}{\bar{U}_\infty} = \frac{5}{\bar{U}_\infty}\sqrt{\left(\frac{\tau_w}{\rho}\right)}$$

and $\quad \tau_w = 0{\cdot}0296\,\rho\,\bar{U}_\infty^2\,Re_x^{-0\cdot2}$.

The final equation can be given several equivalent forms depending on whether the use of the Nusselt or Stanton number is preferred. In terms of the former

* For the simple approach under discussion, it is not surprising that these two methods of fitting give noticeably different matching conditions.

$$Nu_x = \frac{0{\cdot}0292\ Re_x^{0{\cdot}8}\ Pr}{1 + 2{\cdot}12\ Re_x^{-0{\cdot}1}(Pr - 1)}. \quad (12.12)$$

The relation resulting from this equation has been plotted in Fig. 13 from which it is seen that it leads to a virtually linear plot in logarithmic co-ordinates. For this reason it is customary to replace it by the equation

$$Nu_x = \tfrac{1}{2} c_f\ Re_x\ Pr^{1/3} \quad (12.12a)$$

which can be fitted to it on a numerical basis.

It is evident from the preceding remarks that the introduction of τ_w as a function of x into (12.11) is heuristic, and that the final relation cannot apply to Prandtl numbers differing appreciably from unity, or for surfaces other than a flat plate. This is fully confirmed by experiment, and sometimes the factor 2·12 in the denominator of (12.12) is replaced empirically by a function of the Prandtl number to counteract this deficiency to a certain extent.

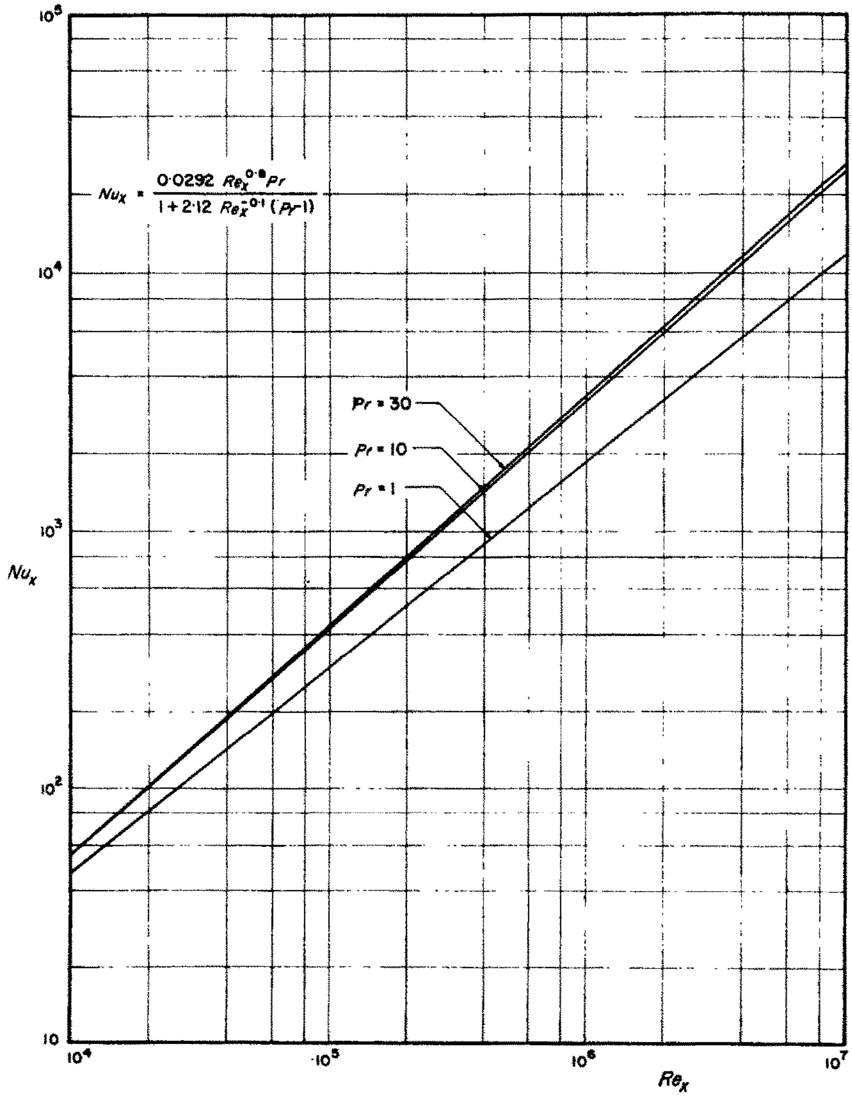

FIG. 13. Variation of Nusselt number with Reynolds and Prandtl numbers on a flat plate. Theory due to Prandtl and Taylor.

The subsequent developments, notably those due to von Kármán [34], Martinelli [57], Rannie [44], Deissler [69, 70, 36, 71, 72, 64, 73], van Driest [65, 35] and Loitsianskii [74], followed the preceding pattern of derivation, but attempted to improve the resulting equations by choosing smoother, and more accurate expressions for the law of the wall. von Kármán's assumptions listed in Table 4 led him to derive the equation

$$Nu_x = \frac{\frac{1}{2}Re_x c_f}{1+5\sqrt{(\frac{1}{2}c_f)}(Pr-1)+\ln[1+5/6(Pr-1)]}. \quad (12.13)$$

For Prandtl numbers close to unity, this expression differs little from the Prandtl–Taylor equation (12.12), as expected. Its range of applicability extends to somewhat higher Prandtl numbers (about thirty), but it also fails to take into account the circumstances which are characteristic of very low Prandtl numbers.

Particularly extensive, and successful within the limitations of the method, were the investigations undertaken by Deissler [69, 70, 36, 71, 72, 64, 73]. The analytic form of the law of the wall used in these calculations has been listed in Table 4. Owing to its complex form, and to the fact that it consists of two different expressions for the ranges $0 \leqslant y^+ \leqslant 26$ and $y^+ > 26$, all integrations must be performed numerically. The assumption of a law of the wall determines implicitly an expression for the eddy viscosity, and this, together with the assumption $Pr_t = 1$, determines an expression for the eddy conductivity. Thus all quantities in (12.2) have been determined by suitable assumptions, and the equation can be integrated. This is best done by first casting it in dimensionless form in terms of the reduced temperature

$$\theta^+ = \frac{\bar{\theta}c_p}{v_*} \cdot \frac{\tau_w}{\dot{q}_w}, \quad (12.14)$$

when it becomes

$$\left(\frac{1}{Pr} + \frac{1}{Pr_t} \cdot \frac{\nu_t}{\nu}\right)\frac{d\theta^+}{dy^+} = 1, \quad (12.15)$$

or alternatively

$$\left(\frac{1}{Pr} + \frac{\epsilon^+ - 1}{Pr_t}\right)\frac{d\theta^+}{dy^+} = 1. \quad (12.15a)$$

Upon integration, this equation leads to a family of universal velocity profiles with the Prandtl number as a parameter. In the range $0 < y^+ < 26$, where the more complex expression is used, see Table 4, the temperature profile is given by the differential equation

$$\left.\begin{aligned}\frac{d\theta^+}{dy^+} = \frac{1}{1/Pr + n^2u^+y^+[1 - \exp(-n^2u^+y^+)]}\\(0 \leqslant y^+ \leqslant 26).\end{aligned}\right\} \quad (12.16)$$

In the range $y^+ > 26$, the Prandtl–Kármán law of the wall is used, $1/Pr$ is neglected with respect to $(\epsilon^+ - 1)/Pr_t$ in (12.15), and the equivalent simplification is made in the expression for the shearing stress. Thus, the resulting expressions

$$u^+ - u_1^+ = \theta^+ - \theta_1^+, \quad (y^+ > 26) \quad (12.17)$$

and

$$u^+ - u_1^+ = \frac{1}{K}\ln\left(\frac{y}{y_1^+}\right), \quad (K = 0{\cdot}36) \quad (12.17a)$$

are entirely equivalent to (12.9) and (12.10) of the elementary Taylor–Prandtl theory described earlier, the difference consisting merely in the choice of the law of the wall (including the choice of $K = 0{\cdot}36$ instead of $K = 0{\cdot}4$) and in the choice of $y_1^+ = 26$ instead of $y^+ = 30$ for the buffer layer in von Kármán's theory.

The universal temperature profiles which result from the numerical integration of (12.16), together with the relation from (12.17) and (12.17a), are shown in Fig. 14. Two curves resulting from the Prandtl–Taylor expression (12.9), namely for $Pr = 30$ and $Pr = 300$, have been added for comparison to illustrate the limitations of (12.12). It is clear that the major difference between the two procedures lies in the choice of the law of the wall in the neighborhood of the wall. Hence, for the same reasons as before, Deissler's theory cannot be valid for very low Prandtl numbers. It has, in fact, been used only for $Pr > 0{\cdot}73$.

Deissler [64] developed his theory to include the compressible boundary layer on a flat plate. There is no difficulty in outlining its application to the present case. It is realized that the universal velocity profile by itself does not

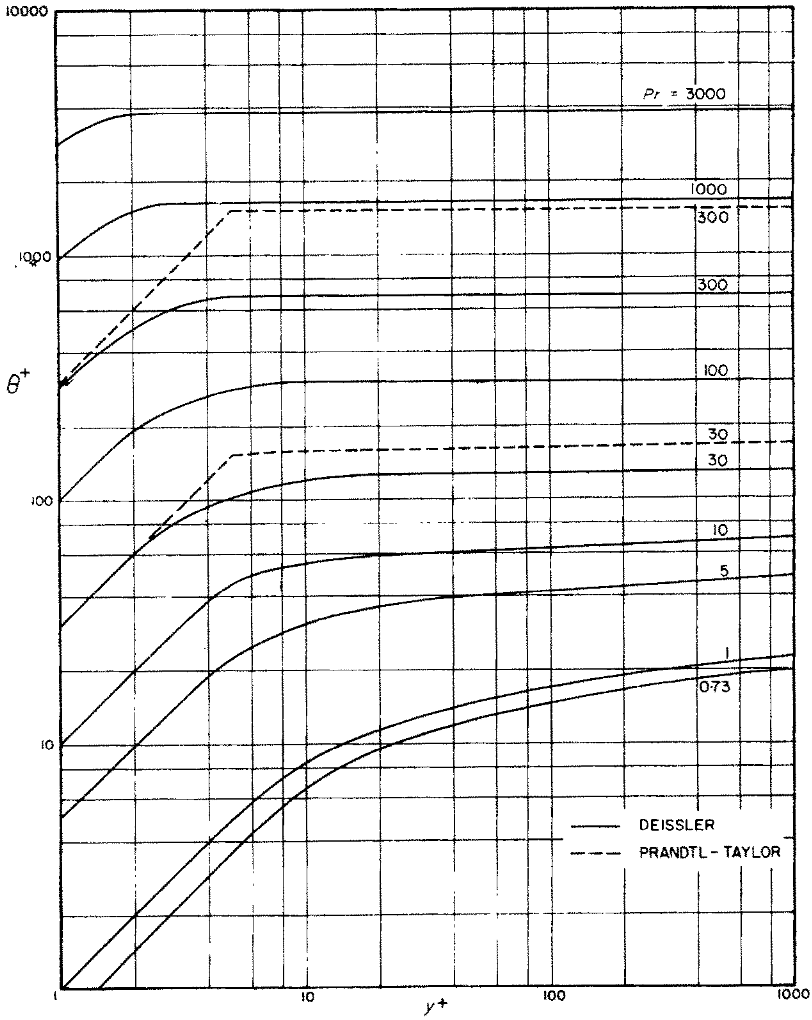

FIG. 14. Universal temperature profiles, after Deissler [36]. Broken curves from (12.9) after Prandtl–Taylor.

represent an actual velocity profile, since it contains the ratio $\tau_w/\dot{q}_w$ as a parameter, (12.14), in analogy with (12.8) and (12.9). In order to develop the theory, it is only necessary to adjust the boundary conditions for temperature in the same way as before, the difference being that now numerical methods must be substituted for explicit formulae. This calculation can be simplified if it is noted that the local Stanton number

$$St = \frac{\dot{q}_w}{c_p \bar{U}_\infty \theta_\infty \rho}$$

can be expressed in terms of the reduced quantities

$$U_\infty^+ = \frac{\bar{U}_\infty}{v_*} \quad \text{and} \quad \theta_\infty^+ = \frac{\theta_\infty c_p}{v_*} \cdot \frac{\tau_w}{\dot{q}_w}$$

as

$$St = \frac{1}{U_\infty^+ \theta_\infty^+}. \tag{12.18}$$

Knowing τ_w it is possible to calculate U_∞^+ and to determine the value of y^+ which corresponds to it. The latter, together with the graphs in Fig. 14, determines θ_∞^+ for given values of θ_∞^+

and Pr, and so the Stanton number. This relation is entirely equivalent to (12.12) or (12.13); the Stanton number was chosen here because it contains no characteristic length, and constitutes a local quantity.

Deissler employed an equivalent procedure for heat transfer in pipes and channels, and the resulting relations led to very good agreement with experimental data for $0{\cdot}73 < Pr < 4000$. The same is also true of the extension to compressible boundary layers on flat plates. Consequently, it might prove useful in boundary layer calculations if an elementary approach is considered to be desirable.

13. EXTENSIONS OF THE ELEMENTARY THEORY

The elementary theory of heat transfer has been successfully extended to the calculation of thermal entry lengths in pipes and channels [70, 75, 76], but no detailed discussion need be given here because a mathematically exact theory is now available, Section 14. A further extension includes the case of variable viscosity [71] which is of interest, since the exact theory has not been developed in sufficient detail, and the elementary theory can serve as an illustration of the trends to be expected where this important phenomenon is taken into account. Owing to the differences in the manner in which the viscosity depends on temperature,* the theory for liquids must be formulated separately from that for gases. The real variation of the viscosity of liquids with temperature is approximated by the empirical relation

$$\frac{\mu}{\mu_w} = (1 - \beta\theta^+)^d \tag{13.1}$$

where

$$\beta = \frac{\dot{q}_w v_*}{c_p T_w \tau_w}. \tag{13.2}$$

In the case of liquids this corresponds to the assumption

$$\frac{\mu}{\mu_w} = \left(\frac{t}{t_w}\right)^d \tag{13.3}$$

* It increases with temperature in gases, but decreases in liquids.

where t is measured with respect to a suitably selected zero (usually 0°F) to ensure a good fit. The exponent d must also be fitted empirically, and for liquids its value ranges from $d = -1$ to $d = -4$. With these assumptions, Deissler's universal temperature and velocity profiles for $y^+ < 26$ now become

$$\frac{du^+}{dy^+} = \frac{1}{(1 - \beta\theta^+)^d + n^2u^+y^+ + [1 - \exp(-n^2u^+y^+/(1 - \beta\theta^+)^d)]} \tag{13.4}$$

and

$$\frac{d\theta^+}{dy^+} = \frac{1}{1/Pr_w + n^2u^+y^+ \{1 - \exp[-n^2u^+y^+/(1 - \beta\theta^+)^d]\}}. \tag{13.5}$$

For larger values of u^+ the terms containing variable viscosity are neglected, and the theory for constant properties in its simplified form, (12.17) and (12.17a) are used. Numerical integrations lead to the universal profiles shown in Figs. 15 and 16 with β as a parameter, $d = -4$ and for $Pr_w = 10$. It is noted that the effect of changing the sign of β ($\beta > 0$ addition of heat, $\beta < 0$ extraction of heat for a liquid) is opposite on velocity and temperature. The universal velocity and temperature profiles can now be utilized to yield relations for the Stanton number, (12.18), in the same manner as was the case with constant viscosity.

The variation of thermal conductivity, viscosity and density with temperature in gases can also be accounted for in the particular case when

$$\frac{k}{k_w} = \frac{\mu}{\mu_w} = \left(\frac{T}{T_w}\right)^d \quad \text{and} \quad \frac{\rho}{\rho_w} = \frac{T}{T_w}$$

(T, T_w absolute temperatures) but for details the reader is referred to the original paper [36].

The success of the elementary theory described in this as well as in Section 12 seems to suggest that under the present circumstances the result might be insensitive to the variation of the ratio $\tau/\dot{q}$ with transverse distance y. This detail is also discussed [72, 64].

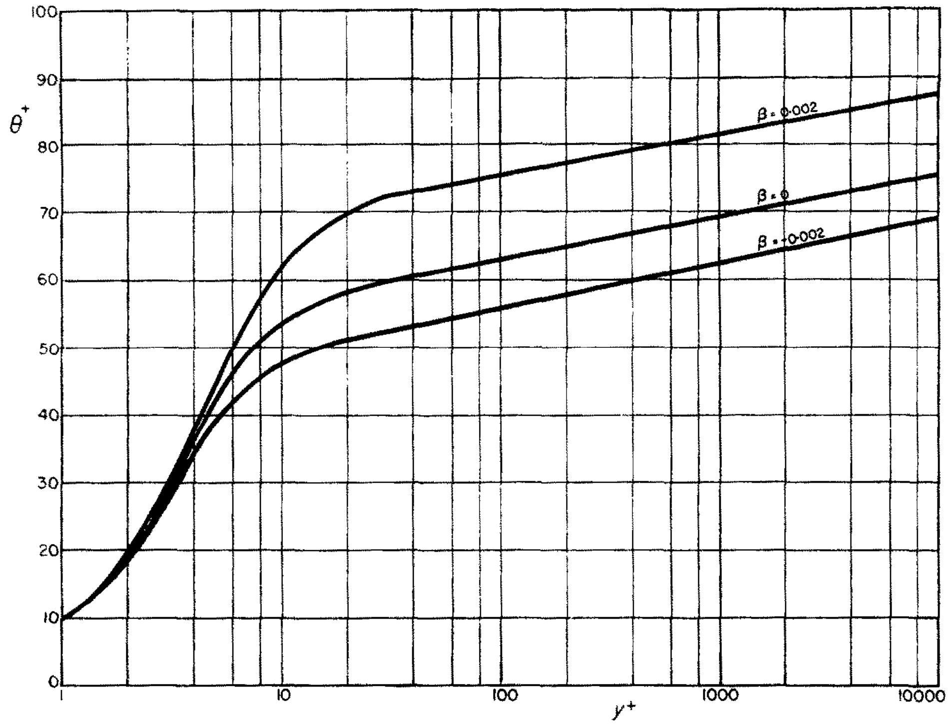

FIG. 15. Deissler's [36] universal temperature profiles, for a liquid with $\frac{\mu}{\mu_w} = \left(\frac{t}{t_w}\right)^{-4}$ t, t_w—reference temperatures ($\beta > 0$ addition of heat, $\beta < 0$ extraction of heat).

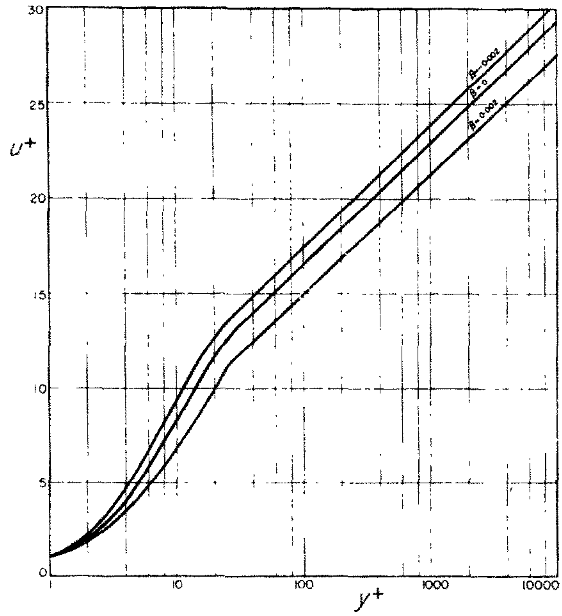

FIG. 16. Deissler's [36] universal velocity profiles for a liquid with $\frac{\mu}{\mu_w} = \left(\frac{t}{t_w}\right)^{-4}$ t, t_w—reference temperatures ($\beta > 0$ addition of heat, $\beta < 0$ extraction of heat).

14. SPALDING'S EXACT THEORY

An exact theory of heat transfer across turbulent boundary layers becomes possible, if it is realized that the von Mises form of the energy equation can be integrated without the need to precede the calculation by an explicit integration of the equation of motion (5.1), when use is made of the law of the wall, (6.2). This circumstance was first recognized by Spalding [2]. From what has been said before, it is clear that such a theory will be exact *only* in the mathematical sense, since the derivation of the differential equations, the explicit expression for the law of the wall to be used, and the relation between the eddy conductivity and eddy diffusivity implied in the assumption concerning the turbulent Prandtl number Pr_t are all of an empirical nature. To be more precise, the expression for the dimensionless effective viscosity ϵ^+ arising from (8.9) and (8.10) is used, it being possible to interpret the derivations in one of two ways. If it is assumed that the law of the wall is exact, then the adoption of (8.9) implies that $\tau \approx \tau_w$ which in turn restricts the result to cases when conditions (12.5a) and (12.5b) are satisfied. Alternatively, it can be asserted that the expression for ϵ^+ in terms of u^+ or y^+ constitutes the starting point of the theory and this would merely imply that the law of the wall is only approximate, a fact which can be said to be consistent with the diagram in Fig. 8.

The crucial transformation consists in introducing the two independent variables, u^+ defined in (6.3), and

$$x^+ = \int_{x_0}^{x} \frac{v_*(x)}{\nu}\,\mathrm{d}x. \qquad (14.1)$$

A scrutiny of the energy equation

$$\frac{\partial\bar\theta}{\partial x} = \frac{\partial}{\partial\psi}\left(a_e\bar u\frac{\partial\bar\theta}{\partial\psi}\right), \quad \bar\theta(x,\psi) \qquad (14.2)$$

in which the temperature difference $\bar\theta$ is expressed as a function of the co-ordinate x and the stream function ψ, reveals that the new variables x^+ and u^+ depend each on one of the old variables only. In particular, as seen from (14.1), x^+ depends on x alone, and it has been shown in Section 8 that u^+ depends on ψ alone. Consequently $\partial x^+/\partial\psi = \partial u^+/\partial x = 0$, and the transformation becomes particularly simple. Noting that $\mathrm{d}x^+/\mathrm{d}x = v_*/\nu$ and that $\mathrm{d}u^+/\mathrm{d}\psi = 1/(\nu u^+\epsilon^+)$, we find that

$$\left(\frac{\partial\bar\theta}{\partial x}\right) = \left(\frac{\partial\bar\theta}{\partial x^+}\right)_u \cdot \frac{v_*}{\nu}$$

$$\text{and}\quad \left(\frac{\partial\bar\theta}{\partial\psi}\right)_x = \left(\frac{\partial\bar\theta}{\partial u^+}\right)_{x^+}\cdot\frac{1}{\nu u^+\epsilon^+}.$$

Inserting these values into the energy equation (14.2), it is easy to verify that it now becomes

$$\frac{\partial\bar\theta}{\partial x^+} = \frac{1}{\epsilon^+u^+}\cdot\frac{\partial}{\partial u^+}\left(\frac{1}{Pr_e}\frac{\partial\bar\theta}{\partial u^+}\right), \qquad (14.3)$$

where $\bar\theta(u^+, x^+)$ is expressed as a function of the reduced co-ordinate x^+ and reduced velocity u^+.

The determining equation (14.3) is a form of the Fourier equation in which the thermal conductivity (replaced here by $1/Pr_e$) appears as a function of position (replaced here by the reduced velocity u^+) and which is multiplied by the variable coefficient $1/(\epsilon^+u^+)$. It can be integrated numerically if proper assumptions are made about the effective Prandtl number and about the law of the wall which specifies ϵ^+ in terms of u^+. The boundary conditions most appropriate for the integration of (14.3) are

$$\left.\begin{aligned} \bar\theta &= 1 \text{ at } x^+ = 0 \text{ and all } u^+ > 0\\ \bar\theta &= 1 \text{ at } u^+ = \infty \text{ and all } x^+ > 0\\ \bar\theta &= 0 \text{ at } u^+ = 0 \text{ and all } x^+ > 0,\end{aligned}\right\} \qquad (14.4)$$

so that now

$$\bar\theta = \frac{T_w - T}{T_w - T_\infty} \qquad (14.5)$$

has been normalized with respect to $T_w - T_\infty$. The boundary conditions are characteristic of the case when a thermal boundary layer begins to develop from $x = x_0$ onwards, Fig. 17. Owing to the fact that the energy equation is linear, the basic solution can be adapted to any set of linear boundary conditions by superposition.

The energy equation in the form (14.3) together with the required physical assumptions and boundary conditions (14.4) defines a universal function

$$\bar\theta(x^+, u^+, Pr), \qquad (14.6)$$

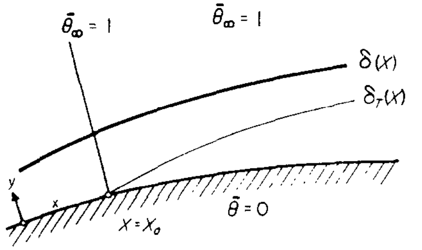

FIG. 17. Boundary conditions for Spalding's energy equation.

since

$$Pr_e = \frac{1 + k_t/k}{Pr + Pr_t(k_t/k)} = \frac{1/Pr + \mu/\mu_t \cdot 1/Pr_t}{1 + \mu_t/\mu} \tag{14.7}$$

contains the Prandtl number within it. As usual, interest is centred only on the derivative

$$Sp(x^+, Pr) = \left(\frac{\partial\theta}{\partial u^+}\right)_{u^+=0}, \tag{14.8}$$

and numerical integration can confine itself to this Spalding function. This is due to the fact that the Stanton number, the skin friction coefficient and the Spalding function are connected by the relation

$$St = Sp(\tfrac{1}{2}c_f)^{1/2}, \tag{14.9}$$

which serves as a basis for the solution of heat transfer problems. It is realized that (14.9) does not itself constitute a solution, and that now, but not before the integration of the differential equation, it is necessary to determine the distribution of the friction velocity v_* or skin-friction coefficient c_f along the wall. This will yield the variation of the Stanton number in terms of x^+, and (14.1) will provide the relation $x^+(x)$, so that the function $St(x)$ can be established.

The task of tabulating the Spalding function for different values of the Prandtl number and for different assumptions concerning the turbulent Prandtl number, Pr_t, is yet to be performed. So far, only the simplest case when

$$Pr_e = 1$$

has been integrated numerically by Spalding [1].* The case when $Pr_e = 1$ includes the one for which $Pr = Pr_t = 1$ simultaneously. The function

$$Sp(x^+, 1) \tag{14.10}$$

is shown plotted as curve (a) in Fig. 18. Spalding's equation (8.4a) for the law of the wall is particularly well-adapted to the form of (14.3) because it leads directly to an expression for ϵ^+ in terms of u^+, (8.5). The present values are somewhat tentative, since the optimum numerical values for the coefficients in (8.4a) are uncertain. The significance of curve (b) marked "Lighthill's solution" will be explained in Section 17. Spalding [1] provided also an approximate, integral, explicit expression for the function (14.10), namely

$$x^+ = 0{\cdot}157\,\delta_T^3 + \frac{1}{72}B^2A\frac{\delta_T^7}{4!} + \dots + \frac{1}{(n+3)^2}\cdot B^{(n+1)}A\frac{\delta_T^{n+3}}{n!} + \dots \tag{14.11}$$

where the thermal boundary layer thickness δ_T is given by

$$\delta_T = \frac{(\frac{1}{2}c_f)^{1/2}}{St}. \tag{14.12}$$

As already mentioned, superposition can be utilized to include problems with complex temperature distributions, and in this connection the papers by Lighthill [77], Tribus and Klein [78], Rubesin [79] and Eckert *et al.* [80] are worthy of attention.

15. EXTENSION OF SPALDING'S THEORY TO AXI-SYMMETRIC BOUNDARY LAYERS

There is no difficulty in extending the method of the preceding section to axi-symmetrical boundary layers by the application of the Mangler transformation [81, 19]. If x, y denote the co-ordinates for the equivalent two-dimensional case, then

$$\tilde{x} = \frac{1}{R^2}\int_0^x r^2(x)\,\mathrm{d}x; \quad \tilde{y} = \frac{r(x)}{R}y \tag{15.1}$$

where $r(x)$ describes the contour, and R is an arbitrary reference length. Correspondingly

* A method of starting the solution from the singular point at $x^+ = 0$ is described in Section 17.

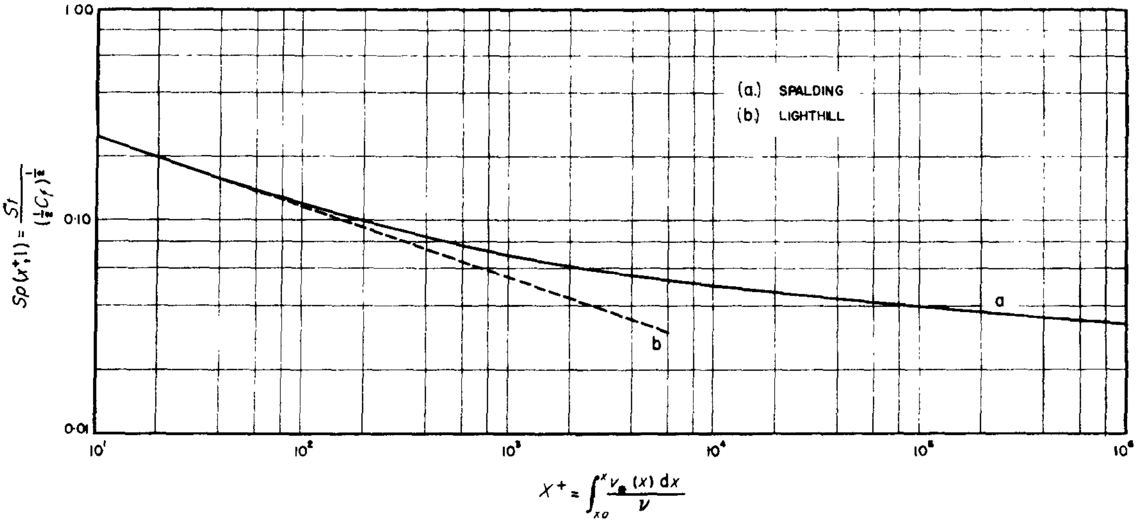

FIG. 18. Spalding's function, $Sp(x^+, 1)$, for $Pr_e = 1$ and for Spalding's function, (8.4), [1].
(a) Spalding's function, (14.10);
(b) Lighthill's solution, (17.13) with $Pr = 1$, [77].

$$\tilde{u} = \bar{u} \quad \text{and} \quad \tilde{v} = \frac{R}{r}\left(\bar{v} + \frac{r'}{r}\, y\, \bar{u}\right),$$

where the prime denotes differentiation with respect to x. Hence

$$\psi = \int_0^y \tilde{u}\, \mathrm{d}\tilde{y} = \frac{R}{r(x)} \int_0^y \bar{u}\, \mathrm{d}y.$$

This proves that it suffices to replace ψ in the equations of the preceding section by

$$\frac{R}{r(x)}\, \psi,$$

so that now

$$\bar{u} = \frac{R}{r}\, \frac{\partial \psi}{\partial y}, \tag{15.2}$$

and the energy equation can be written

$$\frac{\partial \bar{\theta}}{\partial x} = \frac{r(x)}{R} \cdot \frac{\partial}{\partial \psi}\left(a_e \frac{r(x)}{R}\, \bar{u}\, \frac{\partial \bar{\theta}}{\partial \psi}\right), \tag{15.3}$$

which simplifies to the same equation (14.3) as before. The only difference is in the calculation of the $\bar{v}$-component of velocity and, evidently, the computation of $\tau_w(x)$. Otherwise, the numerical results of the preceding section can be transfered directly.

16. BOUNDARY CONDITIONS

The considerations of the preceding sections have resulted in a mathematical formulation of the problem of turbulent convection which is valid within the limitations of the physical assumptions which have led to it. However, the physical problem has not been solved completely owing to the fact that the boundary conditions for (14.2) have not yet been discussed in sufficient detail.

When analysing an equivalent problem in laminar convection, the formulation of the boundary conditions for the energy equation presents little difficulty. In general, as we recall, it is necessary to specify values of the temperature difference θ at the wall ($\theta = 0$ at $y = 0$) and "at infinity" ($\theta = \theta_\infty$ at $y = \infty$), and to prescribe a temperature profile at some cross section $x = x_o$. This is usually determined at the leading edge of the cylinder where one of two conditions prevail. The leading edge may be sharp, and then the physical problem is adequately described by specifying $\theta = \theta_\infty$ at the leading edge (usually $x_o = 0$). Alternatively, the leading edge is blunt, and then the initial temperature profile is that which corresponds to Hiemenz' stagnation flow where Frössling's [82] Blasius series can be used. The simplicity of these conditions is due to the fact that, generally speaking, any boundary layer will at first be laminar. In the case of turbulent convection circumstances become more complex, and a complete and fully satisfactory classification of all important sets of boundary conditions can be obtained only in the course of further research.

The first difficulty arises in connection with the condition "at infinity". In laminar convection it is sufficient to specify $\theta = \theta_\infty$ at $y = \infty$, and the resulting solution for the temperature field $\theta(x, y)$, being of the boundary layer type, determines the thermal boundary layer thickness δ_T. This, in turn, is due to the fact that the condition $u = U(x)$ can also be imposed on the velocity boundary layer at $y = \infty$. In the turbulent convection only a portion of the velocity boundary layer profile is adequately described by the universal law of the wall. It is, nevertheless, used across the whole of the boundary layer to mitigate the mathematical difficulties and in the conviction that the major portion of the temperature drop takes place within it, and that it, consequently, constitutes the major portion of the resistance to heat flow. Moreover, as $y^+ \to \infty$, the velocity $u^+ \to \infty$, and it is necessary to terminate the turbulent boundary layer at a value of u^+ which corresponds to the free-stream velocity $\bar{U}(x)$. Thus the transition from $\bar{u} = 0$ at the wall to $\bar{u} = \bar{U}(x)$ at $y > \delta$ is not smooth and asymptotic. As a result, it is necessary to specify the boundary condition for temperature in relation to this somewhat artificial edge of the boundary layer. As long as the thermal boundary layer thickness δ_T is smaller than this artificial velocity boundary layer thickness δ, the resulting solution constitutes a good approximation, it being assumed that

$$\bar{\theta} = \bar{\theta}_\infty \text{ at } y = \delta \text{ or at } \bar{u} = \bar{U}(x),$$

see Fig. 19a. Then, the boundary layer character of the solution for $\bar{\theta}(x)$ asserts itself, and the resulting temperature profile adequately determines the thermal boundary layer thickness. Such conditions prevail for all values of the Prandtl number near a point where the thermal boundary begins to develop, on condition that it does so within a fully developed, turbulent velocity boundary layer for large values of the

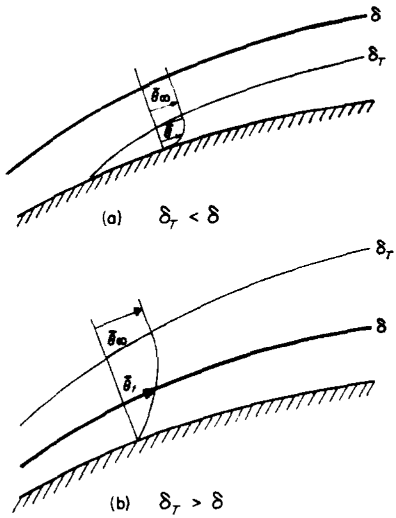

FIG. 19. Temperature boundary condition (a) $\delta_T < \delta$; (b) $\delta_T > \delta$.

Prandtl number, since then $\delta_T < \delta$ at any position x. This circumstance explains why, in general, the elementary theories of turbulent convection, Section 12, have been much more successful for very high than for very low Prandtl numbers.

The approximation is still an acceptable one in the case of gases, when the Prandtl number does not differ much from unity, and when Spalding's heuristic approximation

$$Sp(x^+, Pr) = Sp(x^+, 1) \,.\, Pr^{-2/3} \qquad (16.1)$$

can render good service.

For low Prandtl numbers, circumstances are reversed, and, except for a short distance from a step in temperature, the thermal boundary layer continues to be much larger than the velocity boundary layer, Fig. 19b. The proper value, $\bar{\theta}_t$ in Fig. 19b, which exists at $y = \delta$ cannot be determined and the method fails, particularly for extremely low Prandtl numbers, since the difference between $\bar{\theta}_\infty$ and $\bar{\theta}_t$ can become very appreciable indeed. This set of circumstances, as is known [57], caused difficulties in the adaptation of the elementary theories of heat transfer to such very low Prandtl numbers as are characteristic of molten metals.

It seems that two ways are open for further research in this respect. First, it might be necessary to include an expression for the law of the wake, particularly at high Reynolds numbers and at very low Prandtl numbers, since a large proportion of the temperature drop occurs across the wake-like zone of the boundary layer. This course is fraught with mathematical difficulties.

The conditions at $x = x_0$ are also difficult to assess with certainty. It is known from the researches of Prandtl [83, 16] that a turbulent velocity boundary layer on a flat plate grows approximately as if it had been started at the leading edge, say by tripping, Fig. 20. Thus, at a distance x from the leading edge, and in the fully turbulent region, the velocity profile in a turbulent boundary layer which had passed through a laminar range and through the transition zone is the same as that which would start by being turbulent at the leading edge itself. It is possible to conjecture, but it is not certain, that the same applies to the temperature profile in cases when it starts to grow from a sharp leading edge. It is clear, however, that in most cases of practical interest, particularly in the case of blunt leading edges, the development of the temperature profile must be traced through the laminar region and through the transition zone. It is at present not clear what simplifications, if any, can be made in this respect.

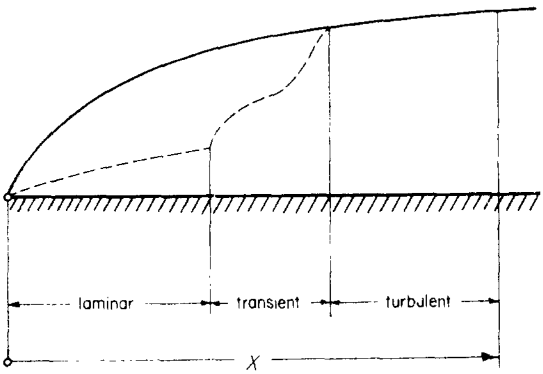

FIG. 20. Growth of a turbulent boundary layer on a flat plate.

* More precise experiments seem to indicate that this virtual starting point is located somewhat further downstream.

17. TWO LIMITING CASES

There exist two limiting cases of turbulent convection which can be solved from first principles and which are of some importance in their applications.

(a) When the Prandtl number becomes very small, as is the case with molten metals, the greatest part of the temperature drop, Fig. 19b, occurs outside the velocity boundary layer, and in the case when $Pr \to 0$ it is seen that $\bar\theta_l \to 0$. Hence it is possible to assume in the energy equation (5.3) that $\bar u = \bar U(x)$ with

$$\bar v = -\frac{\partial \bar U}{\partial x} y,$$

and the equation can be written

$$\bar U \frac{\partial \bar\theta}{\partial x} - \frac{\partial \bar U}{\partial x} y \frac{\partial \bar\theta}{\partial y} = \frac{\partial}{\partial y}\left[\left(k + \frac{\nu_t c_p}{Pr_t}\right)\frac{\partial \bar\theta}{\partial y}\right]. \quad (17.1)$$

Since the process of heat transfer occurs across the wake-like turbulent zone, it is further permissible to assume that $\nu_t \ll k Pr_t / c_p$ and the resulting, simple equation

$$\bar U \frac{\partial \bar\theta}{\partial x} - \frac{\partial \bar U}{\partial x} y \frac{\partial \bar\theta}{\partial y} = a \frac{\partial^2 \bar\theta}{\partial y^2} \quad (17.2)$$

can be solved, particularly when $\bar U(x) = \bar U_\infty$ = constant. Its form is then

$$\frac{\partial \bar\theta}{\partial x} = \frac{a}{\bar U_\infty}\left(\frac{\partial^2 \bar\theta}{\partial y^2}\right). \quad (17.3)$$

This is the Fourier equation in two dimensions for which many solutions exist [97] in the theory of heat conduction. These can be readily adapted to the preceding case of so-called "slug flow". The theory of slug flow has been developed systematically [69, 84, 60, 85, 57, 61]. Experimental measurements for low Prandtl numbers have also been reported [12, 86, 85, 87]. In the case of a flat plate, the boundary conditions are:

$$\bar\theta = \bar\theta_\infty \text{ at } y = \infty \text{ for all } x > 0$$
$$\bar\theta = \bar\theta_\infty \text{ at } x = 0 \text{ for all } y > 0$$
$$\bar\theta = 0 \text{ at } y = 0 \text{ for all } x > 0$$

and the solution is

$$\bar\theta = \bar\theta_\infty \operatorname{erf}\left[\frac{y}{2[(a/\bar U_\infty)x]^{1/2}}\right]. \quad (17.4)$$

Slug flow solutions constitute solutions limiting case when $Pr \to 0$. When the Prandtl number is not extremely low, the layer adjacent to the wall will provide an additional resistance to heat flow which can be accounted for by noting that the temperature profile implied in (17.2) still constitutes a very good approximation. It is then possible to determine the temperature $\bar{\theta}_t$, Fig. 19b, at the edge of the velocity boundary layer and to improve the approximation by re-computing the rate of heat transfer with the aid of the exact theory and the proper boundary conditions. At present this cannot be carried out owing to the absence of tables of the Spalding function for very low Prandtl numbers.

(b) The second limiting case occurs for $Pr \to \infty$, when most, and in the limit, all of the temperature drop takes place within the laminar sublayer. It is thus possible to obtain the solution for turbulent convection by the use of the energy equation for laminar flow, together with the information that the mean velocity profile in the laminar sublayer is linear. Since

$$\tau_w = \mu \frac{\partial u}{\partial y} = \text{constant}$$

we have

$$u = \frac{\tau_w}{\mu} y$$

and the equation of continuity leads to

$$v = -\frac{1}{2\mu} \frac{d\tau_w}{dx} y^2.$$

Inserting these values into the heat energy equation (5.3) with $\dot{q}_t = 0$, we obtain

$$\frac{\tau_w(x)}{\mu} y \frac{\partial \theta}{\partial x} - \frac{1}{2\mu} \frac{d\tau_w}{dx} y^2, \qquad \frac{\partial \theta}{\partial y} = a \frac{\partial^2 \theta}{\partial y^2}. \quad (17.5)$$

Readers familiar with the theory of heat transfer through compressible boundary layers will recognize that this equation, in von Mises' form, was solved by Lighthill [77] for the incompressible case by the use of operational methods, and for the case when the wall temperature distribution is arbitrary. There is no difficulty in indicating an elementary derivation of the solution for a constant wall temperature applied from $x = x_o$ onwards, when the boundary conditions are:

$$\left.\begin{aligned} \theta &= \theta_\infty \text{ at } x = 0 \text{ and all } y > 0 \\ \theta &= \theta_\infty \text{ at } y = \infty \text{ and all } x > 0 \\ \theta &= 0 \quad \text{at } y = 0 \text{ and all } x > 0. \end{aligned}\right\} \quad (17.6)$$

The solution must be of the self-similar type, there being no length parameter in the boundary conditions, but the appropriate similarity parameter is not immediately apparent. However, it can be shown [88], or verified, that the substitution

$$\eta = \frac{y^3(\tau_w/\mu)^{3/2}}{9a \int_{x_o}^{x} (\tau/\mu)^{1/2}\, dx} \quad (17.7)^*$$

transforms (17.5) into an ordinary differential equation for $\bar{\theta}(\eta)$, namely

$$\frac{d^2\theta}{d\eta^2} + (\eta + 2/3) \frac{d\theta}{d\eta} = 0, \quad (17.8)$$

which can be solved by elementary methods. The boundary conditions are now

$$\left.\begin{aligned} \theta &= \theta_\infty \text{ at } x = x_o \text{ and all } y > 0 \text{ or } \eta = \infty \\ \theta &= \theta_\infty \text{ at } y = \infty \text{ and all } x > 0 \text{ or } \eta = \infty \\ \theta &= 0 \ \text{ at } y = 0 \ \text{ and all } x > 0 \text{ or } \eta = 0. \end{aligned}\right\} (17.9)$$

The first integral of (17.8) is

$$\frac{d\theta}{d\eta} = C_1 \eta^{-2/3} \exp(-\eta),$$

so that the general solution is

$$\theta(\eta) = C_2 + C_1 \gamma \left(\frac{1}{3}, \eta\right), \quad (17.10)$$

where $\gamma(a, x)$ is the incomplete gamma function [89, 90]

$$\gamma(a, x) = \int_0^x e^{-t}\, t^{a-1}\, dt.$$

Noting that $\gamma(a, \infty) = \Gamma(a)$ and $\gamma(a, 0) = 0$, and taking into account the boundary conditions (17.9), it is now possible to write down the complete, exact solution for our problem:

$$\theta(\eta) = \frac{\theta_\infty}{\Gamma(1/3)} \gamma \left(\frac{1}{3}, \eta\right). \quad (17.11)$$

* A useful alternative form is $\eta = (y^+)^3 Pr / 9x^+$.

In order to calculate the temperature gradient at the wall, it is necessary to determine the derivative

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \left[\left(\frac{\partial\theta}{\partial\eta}\right)\cdot\left(\frac{\partial\eta}{\partial y}\right)\right]_{y=0}. \quad (17.12)$$

In turn, in order to determine the derivative of the incomplete gamma function it is best to consider the series expansion [89, 90]

$$\gamma\left(\frac{1}{3},\eta\right) = 3\eta^{1/3}\left(1 - \frac{1}{4}\eta + \frac{1}{14}\eta^2 + \ldots\right),$$

it being important to realize that

$$[(\partial\theta/\partial\eta)_{\eta=0}] = \infty,$$

whereas the product (17.12) is non-singular. In this manner, it can be shown that

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \frac{\theta_\infty}{\Gamma(4/3)}\cdot\frac{\sqrt{(\tau_w/\mu)}}{[9a\int_{x_o}^{x}\sqrt{(\tau_w/\mu)}\,\mathrm{d}x]^{1/3}}.$$

It is convenient to re-arrange this equation so as to introduce the friction velocity $v_* = \sqrt{(\tau_w/\rho)}$ and the Prandtl number $Pr = \nu/a$, when the form

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \frac{\theta_\infty Pr^{1/3}}{9^{1/3}\Gamma(4/3)}\cdot\frac{v_*/\nu}{[\int_{x_o}^{x}(v_*/\nu)\,\mathrm{d}x]^{1/3}}$$

is obtained. In order to provide a basis of comparison with Spalding's theory, Section 14, it is necessary to introduce the Stanton number, (10.5), and to notice that $v_* = \bar{U}_\infty\sqrt{(\frac{1}{2}c_f)}$. Substituting, further, the numerical constant

$$\frac{1}{9^{1/3}\Gamma(4/3)} = 0{\cdot}53835,$$

we obtain finally

$$\frac{St}{(\frac{1}{2}c_f)^{1/2}} = \frac{0{\cdot}53835}{(x^+)^{1/3}}\cdot Pr^{-2/3}, \quad (17.13)$$

where x^+ has been defined in (14.1).

The asymptotic solution (17.13) provides a good approximation for very large Prandtl numbers, when it can be used directly. Furthermore, as x^+ becomes closer and closer to zero, the present theory merges with the exact theory of Section 14 because any thermal boundary layer will develop first through the laminar sublayer. The diagram in Fig. 18 shows that this is the case for $Pr = 1$, when (17.13) can be used directly up to $x^+ = 100$ or so, depending on the required accuracy. Since (14.3) is singular at $x^+ = 0$, the numerical solution for any Prandtl number can be started with (17.13) and then continued step-by-step.

For the sake of completeness it is useful to write down the formula for the local Nusselt number

$$Nu_x = \frac{\dot{q}_w(x - x_o)}{k\theta_\infty} = 0{\cdot}53835\,Re_x(Pr/x^+)^{1/3} \quad (17.14)$$

where

$$Re_x = \frac{\bar{U}_\infty(x - x_o)}{\nu}$$

and for the mean Nusselt number

$$Nu = \frac{\int_{x_o}^{x}\dot{q}_w(x)\,\mathrm{d}x}{k\theta_\infty} = 0{\cdot}80743\,Pr^{1/3}(l^+)^{2/3} \quad (17.14a)$$

where

$$l = x - x_o \quad \text{and} \quad l^+ = \int_0^l [v_*(x)/\nu]\,\mathrm{d}x.$$

More explicit formulae for a flat plate at zero incidence will be given in another paper [88].

18. ROUGH SURFACES

All preceding considerations have been restricted to the discussion of smooth surfaces. The effects introduced by roughness are of great practical importance, because roughness increases the rate of heat transfer, and because most practical surfaces are rough or develop roughness with use. In addition, the modifications of the flow and temperature fields introduced by roughness occur mainly in or near the laminar sublayer, and their study might contribute to our understanding of the nature of laminar sublayers. On the other hand, the introduction of roughness elements seriously complicates our problem. For this reason, we shall confine ourselves to a few brief remarks, pointing out that an extensive discussion and a comprehensive list of references have been provided by Nunner [14] in his authoritative paper on this subject.

It is superfluous to remark that with very few exceptions, e.g. [10, 11], most measurements involving rough surfaces have been restricted to pipe flow. The very extensive measurements on air flowing through rough pipes performed by Nunner revealed the interesting fact that the temperature profile remains virtually unaffected by the presence of roughness, in remarkable contrast with the effect of the latter on the velocity profile. It was further established that in rough pipes the relation between the temperature ratio $\bar{\theta}/\bar{\theta}_0$ (where $\bar{\theta}_0$ denotes the temperature difference along the center-line) and the velocity ratio $\bar{u}/\bar{u}_0$ is very nearly the same as that for different Prandtl numbers in smooth pipes, increasing roughness corresponding to increasing values of Prandtl number. The only difference consists in the fact that an increase in Prandtl number leaves the velocity profile unaffected, but modifies the temperature profile, the effect of roughness being opposite. From this it can be surmized that the presence of roughness elements gives rise to form drag by introducing numerous wakes into the boundary layer, thereby modifying the velocity profile. The temperature profile, whose shape is largely determined by the temperature gradient in the laminar sublayer remains, therefore, relatively unchanged. The increased turbulence caused by the roughness elements merely reduces the resistance of heat flow through the turbulent layer and exerts a minor influence on the temperature profile.

Bearing these facts in mind, Nunner applied Prandtl's elementary theory (see Section 12) to the calculation of heat transfer rates. He sharply divided the boundary layer into a laminar sublayer, followed by a wake-layer created by the roughness elements, and by the usual turbulent core. Assuming that the shearing stress across the wake-layer varies from that characteristic of rough flow, $\tau_r = \frac{1}{2}c_{fr}\rho\bar{U}_\infty^2$, on the side of the turbulent core, to a value $\tau = \frac{1}{2}c_f\rho\bar{U}_\infty^2$ characteristic of a smooth wall at the same Reynolds number on the side adjacent to the laminar sublayer, he was able to replace (12.11) by

$$Nu_x = \frac{\frac{1}{2}c_{fr}\, Re_x\, Pr}{1 + \bar{u}_l/\bar{U}_\infty\,(c_{fr}/c_f\, Pr - 1}. \qquad (18.1)$$

when $Pr_t = 1$ is assumed. If a different but constant value of Pr_t is thought to be more appropriate, the Prandtl number Pr should be replaced by the ratio Pr/Pr_t. Equation (18.1) was derived by Nunner for the case of a pipe, and has been rewritten here in an equivalent form applicable to a flat plate; it can be transformed in the same way as (12.1) in Section 12. A comparison with experiment shows that the working formulae which follow from (18.1) are reasonable for $Pr \approx 1$, but fail already at $Pr = 7$, which is not surprising.

An attempt to provide a more exact theory for rough surfaces could be based on van Driest's [35] law of the wall which assumes the particularly simple form

$$\epsilon^+ = \tfrac{1}{2}[1 + (1 + 0{\cdot}64y^{+2})^{\frac{1}{2}}] \qquad (18.2)$$

for completely rough walls. For a given relative roughness $k^+ = v_* k/\nu$, where k is the height of a roughness element, the expression

$$\epsilon^+ = \tfrac{1}{2}\,\{1 + [1 + 4K^2y^{+2}\, 1 - \exp(-y^+/26) + \exp(-60y^+/26k^+)^2]^{1/2}\} \qquad (18.3)$$

with $K = 0{\cdot}4$ might be used. A complete solution would entail the skillful elimination of y^+ in favor of u^+ in (18.2) and (18.3) and an integration of (14.3).

19. OUTLINE OF MALKUS' THEORY OF TURBULENT FLOW AND FREE CONVECTION

In the last few years, an attempt has been made to analyse turbulent convection upon quite different lines from those described above. Instead of using the Reynolds equations (whose appropriateness is not beyond dispute) and then postulating reasonably credible mechanisms to account for the observed velocity, friction and heat transfer measurements, this analysis starts with the complete and unaveraged Navier–Stokes equations and introduces some postulates of a physical character, from which velocity distributions and transport rates have been derived without introducing any empirical information. The approach under discussion is made with the understanding that previous semi-empirical approaches possess unsatisfactory features in that they are not general enough to cover many different problems. However, any more general theory may be expected to have greater mathematical complexity, and initial attempts are best made with problems having simple initial and boundary conditions. No attempt has yet been made to apply the approach directly to the central problem of incompressible turbulent boundary layer convection.

Two specific cases were investigated by Malkus: the Rayleigh horizontal heated plate problem [91, 92], and flow between two parallel surfaces [93]. For both problems Malkus investigated the stability of the mean flow, and considered the velocity as a Fourier sine series with the primary argument based upon the separation distance of the surfaces which form the boundaries of the problems. For the Rayleigh problem the temperature field was represented in a similar way. Malkus then introduced the assumption that the series for the local convective transport of heat or momentum terminates at some finite harmonic of the primary argument. This finite harmonic value is interpreted with the aid of the stability equation to correspond to a minimum effective eddy size. This is taken to indicate that all "eddies" down to some minimum size make an effective contribution to the transport of heat, whilst all eddies of smaller size are essentially dissipative and contribute negligibly to transport. Good correspondence with experiment is found for net heat transfer in the Rayleigh problem. For the case of flow between parallel planes, Malkus observed that the "scale" of motion in the solution of the mean-flow stability equation is determined by the imaginary portion of the derivative of the stream function, the largest value of which occurs in the laminar sublayer adjacent to the boundary. From this it is clear that the mean-flow stability characteristics of the laminar sub-layer determine the smallest effective "eddy" size, and thereby determine the characteristic energy dissipation rate. In this view, the laminar sublayer contains a mean flow which is stable at very high Reynolds numbers, and which permits oscillations of finite amplitude down to some scale which is related to the characteristic minimum eddy size. Thus the laminar sublayer is recognized as playing a specific critical part in determining the characteristics of the turbulent layer.

Using a variational method, and introducing some mathematical approximations, it is possible to derive the logarithmic velocity law, the appropriateness of which has been amply demonstrated by experiment. Malkus evaluated the von Kármán constant and found excellent agreement with that measured by Laufer [41]; the "logarithmic intercept", however, did not correspond well, being only about 0·6 of the experimental value.

Further natural convection experiments [94] have shown good agreement with Malkus' theory, and the existence of an apparent relation between laminar instability parameters and turbulent heat transfer has been noticed for certain other flows [95]. For larger physical scales of motion, comparisons have been made in meteorology, e.g. [83], whilst for astrophysics a modified analysis has been presented [96] in which an allowance is made for non-linear interactions between modes, a detail which has not been studied explicitly in Malkus' work.

20. CONCLUDING REMARKS

In the preceding sections we have attempted to review the present understanding of heat transfer across incompressible, turbulent boundary layers and to formulate a consistent method of analysis based upon it. The aim has been to reduce the problem to the solution of a differential equation, as is done in the case of laminar flow. In turbulent flow, the derivation of this fundamental equation cannot be carried out without making a number of empirical assumptions, and in developing this review it becomes clear that there remain several important questions which are, as yet, unanswered. These questions are of critical importance and must be resolved before this theory can be accepted as a definitive one. In this section we propose to review once again these outstanding problems.

The development of the averaged differential (Reynolds) equations as described here is predicated upon a number of assumptions. The assumption of constant properties is essentially one which limits application to small temperature differences, but no investigation has been made which gives an understanding of what can be considered in practice as small temperature differences for various fluids. We may note that such a study is yet to be made not only for turbulent flows, but also for laminar flows. The assumptions regarding the spectral similarities of velocity and temperature fluctuations need to be studied, as do the assumptions regarding the fluctuation correlations, e.g., $\overline{\theta' u'}$. No convincing proof has been offered which would show that the Boussinesq assumption is consistent with the nature of the fluctuation correlations.

The semi-empirical theories will make use, in one way or another, of the universal velocity profile. The extent to which this profile is altered by temperature differences, exerting their influences through their effect on the fluid properties, has not been examined systematically. Knowledge of the extent of such an effect is essential before confidence can be developed in the application of the semi-empirical theories to many cases of practical importance, especially those involving liquids. For cases where temperature differences cause a large variation of kinematic viscosity, it is not known how to form the reduced quantities u^+ and y^+. In particular, it is not known whether local values of v_* and ν, or the values at the wall, should be used.

The universal velocity profile is not essentially a velocity profile, for there is no universal value of u^+ (or correspondingly of y^+) to which it can be asserted that the external stream velocity should be matched. The range of u^+ for which the law can represent boundary layer velocities depends in some way upon parameters such as the Reynolds number and the roughness. In reality, the universal velocity profile serves as an empirical statement of the variation of the dimensionless effective viscosity ϵ^+ with the flow parameters. As stated in Section 8, it is possible to adopt one of two points of view. First the theory can be formulated on the assumption that there exists a universal form for ϵ^+ which is valid in all layers, except the outer, wake-like region. In this case all that is necessary is to determine which one of the proposed analytic expressions provides the best agreement with experimental data. Alternatively it can be asserted that the universal velocity profile constitutes the primary datum. It then becomes necessary to assume that $\tau = \tau_w$ in order to complete the theory and the question arises as to how well and in what circumstances this assumption is borne out by experiment, and whether, perhaps, the final result is simply insensitive to transverse variations in τ. In the first case, the existence of a single universal law of the wall would have to be considered as an acceptable approximation and it will be found that it may vary somewhat with x, that is, essentially, with the Reynolds number.

It is, further, apparent that for very high Reynolds numbers, and for low Prandtl numbers, the wake region becomes significant in determining the convective behavior of the flow. No extensive investigations have been made of this.

The present confusion and lack of agreement about the behavior of the turbulent Prandtl number is highly unsatisfactory, particularly as it affects the application of the semi-empirical theories.

We have indicated the connection between the Navier–Stokes and energy equations and the corresponding Reynolds and averaged energy equations and their boundary layer forms, without, however, being able to indicate a rigorous derivation or to provide evidence of the time-dependent similarities within the flow or at the boundaries. In this connection it has been discovered that this assumption of spectral similarity between the velocity fluctuations u' and the temperature fluctuations θ', which is necessary to assure the similarity of the average profiles $\bar{u}(y)$ and $\bar{\theta}(y)$, seems to be related to the assumption that $Pr_t = 1$ in the Boussinesq formulation of the problem. It would be interesting to explore whether the apparent equivalence of these two alternative assumptions is accidental or whether it has a basis in physical fact. An attempt has been made to indicate the relationship between the laminar sublayer on the one hand and the laminar boundary layer under a turbulent free stream on the other. The characteristics which mutually relate these flows do not appear to have been generally recognized; and, correspondingly, the implications of the stable oscillating character of the flow have not been fully explored in either case. The importance of events within the laminar sublayer to the phenomena in the regions of developed turbulent flow is indicated not only by the discussion in Section 7, but also in Malkus' analysis of turbulent shear flow, Section 19. From these considerations, it is clear that laminar sublayers and laminar boundary layers associated with turbulent free streams deserve careful investigation.

ACKNOWLEDGEMENTS

The authors wish to thank Professors A. Favre and H. Schlichting who, by inviting them to deliver this lecture in Section IV, encouraged them to attempt this survey. Particular gratitude is due to Prof. D. B. Spalding who put at their disposal two unpublished papers without which the story would have no happy ending and who patiently answered many tedious letters. Professors L. N. Persen and R. A. Dobbins were always willing to serve as interested sounding boards; Prof. L. N. Persen checked the derivations in Sections 14 and 15 and solved *de novo* M. J. Lighthill's problem described in Section 17. Mr. T. B. Shen helped with the calculations and the preparation of the drawings.

The work of preparing the lecture is part of the research programme on heat transfer of unsteady flows of the Aeronautical Research Laboratories, Office of Aerospace Research of the United States Air Force whose support is acknowledged. This work was done under Contract No. AF 33(616) 7749.

A grant from the National Science Foundation in Washington, D.C. enabled one of the authors to participate in the Colloquium.

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Zusammenfassung—Für den Wärmeübergang in turbulenter Grenzschicht bei erzwungener Konvektion wird hier ein Überblick über den gegenwärtigen Stand der Kenntnisse vermittelt. Die Grundlagen der halbempirischen Theorie werden von Anfang an nachgeprüft und ihre Grenzen sorgfältig aufgezeigt. Elementare Theorien sind beschrieben und eine Zusammenfassung der mathematisch exakten Theorie von D. B. Spalding ist angegeben. Die Grenzfälle sehr hoher und sehr niedriger Prandtlzahlen werden diskutiert. Auch ist die Theorie der turbulenten Prozesse von W. V. R. Malkus zusammengefasst angeführt. Eine gewissenhafte Untersuchung galt der Klärung aller physikalischen Annahmen, um die dringlich zu beachtenden Grundprobleme und dei Erweiterungsmöglichkeiten der halbempirischen Theorie angeben zu können.

Аннотация—Статья содержит обзор современного состояния в области переноса тепла вынужденной конвекцией в несжимаемом турбулентном пограничном слое. Рассматриваются основы полуэмпирической теории и указываются ее ограничения. Описаны элементарные теории и дается краткое изложение математически строгой теории Д Б. Сполдинга. Рассматриваются частные случаи очень больших и очень малых чисел Прандтля. Вкратце излагается теория турбулентных процессов, предложенная У. В. Р. Малкусом.

Сделана попытка объяснить физические допущения, сформулировать основные проблемы, требующие особого внимания и наметить направление развития полуэмпирической теории.